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1-homogeneous, pseudo-1-homogeneous, and 1-thin distance-regular graphs

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Abstract

Let Γ denote a distance-regular graph with diameter $d \geqslant 2$, and fix a vertex x of Γ. Γ is said to be *1-homogeneous* (resp. *pseudo-1-homogeneous*) *with respect to* x whenever for all integers h and i between 0 and d, inclusive (resp. for all integers h between 0 and $d-1$ and i between 0 and d, inclusive) and for all vertices y and z of Γ with $\partial(x, y) = h$, $\partial(y, z) = i$, $\partial(z, x) = 1$, the number of vertices w of Γ with $\partial(x, w) = j$, $\partial(y, w) = 1$, $\partial(z, w) = k$ is independent of y and z for all j, k $(0 \leqslant j,\ k \leqslant d)$. We characterize these properties algebraically.

The *Terwilliger algebra* $\mathcal{T} = \mathcal{T}(x)$ *of* Γ *with respect to* x is the matrix subalgebra generated by A, $E_0^*, E_1^*, \ldots, E_d^*$, where A is the adjacency matrix of Γ and E_i^* is the diagonal matrix whose nonzero entries are ones in the (y, y) positions for those vertices y such that $\partial(x, y) = i$. Our results concern the left ideal $\mathcal{T}E_1^*$ of $\mathcal{T}$ generated by E_1^*. We show that Γ is 1-homogeneous with respect to x if and only if $\dim E_i^*\mathcal{T}E_1^* \leqslant 3$ $(1 \leqslant i \leqslant d-1)$ and $\dim E_d^*\mathcal{T}E_1^* \leqslant 2$. We also show that when the intersection number $a_1 \neq 0$, Γ is pseudo-1-homogeneous with respect to x if and only if $\dim E_i^*\mathcal{T}E_1^* \leqslant 3$ $(1 \leqslant i \leqslant d)$. We then characterize these properties according to the structure of the summands in the decomposition of $\mathcal{T}E_1^*$ into minimal left ideals.

Finally, we use these decompositions to describe a related family of distance-regular graphs. Let $\mathcal{L}$ denote a minimal left ideal of $\mathcal{T}$. Then $\mathcal{L}$ is said to be *thin* if $\dim E_i^*\mathcal{L} \leqslant 1$ $(0 \leqslant i \leqslant d)$. The *endpoint* of $\mathcal{L}$ is $\min\{i \mid E_i^*\mathcal{L} \neq 0\}$. The graph Γ is said to be *1-thin with respect to* x when every minimal left ideal of $\mathcal{T}$ with endpoint 1 is thin. It is known that Γ is 1-thin with respect to x with a unique minimal left ideal of endpoint 1 if and only if Γ is bipartite or almost bipartite (in either case Γ is 1-homogeneous with respect to x). We show that Γ is 1-thin with respect to x with exactly two

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minimal left ideals of endpoint 1 if and only if Γ is pseudo-1-homogeneous with respect to x and the intersection number a_1 is nonzero.

Keywords: Terwilliger algebra

1. Introduction

Let Γ denote a finite undirected connected graph with vertex set X, distance-function ∂, and diameter d. Recall that Γ is *distance-regular* whenever for all integers h, i, j $(0 \leqslant h, i, j \leqslant d)$ there are *intersection numbers* p^h_{ij} such that for all vertices x, y of Γ with $\partial(x, y) = h$, the number $|\{z \in X \mid \partial(x, z) = i, \ \partial(y, z) = j\}| = p^h_{ij}$. For the rest of this section, assume that Γ is distance-regular. We consider some algebras associated with Γ (see Section 2). Fix a vertex x of Γ. The Bose–Mesner algebra $\mathcal{M}$ of Γ is the matrix subalgebra generated by the adjacency matrix A of Γ and the dual Bose–Mesner algebra $\mathcal{M}^* = \mathcal{M}^*(x)$ of Γ with respect to x is the diagonal matrix subalgebra generated by the dual idempotents $E^*_0, E^*_1, \ldots, E^*_d$ of Γ with respect to x. The Terwilliger algebra $\mathcal{T} = \mathcal{T}(x)$ of Γ with respect to a vertex x is the matrix subalgebra generated by $\mathcal{M}$ and $\mathcal{M}^*(x)$.

In this paper we relate certain algebraic restrictions on $\mathcal{T}$ and certain combinatorial regularity conditions on Γ. We begin by recalling some of the regularity conditions which arise in our work. The graph Γ is *1-homogeneous* whenever for all integers h, i, j, k, $(0 \leqslant h, i, j, k \leqslant d)$ there is a structure constant $\gamma^{h,i}_{j,k}$ such that for all vertices x, y, z of Γ with $\partial(x, y) = h, \partial(y, z) = i, \partial(z, x) = 1$, the number $|\{w \in X \mid \partial(x, w) = j, \ \partial(y, w) = 1, \ \partial(z, w) = k\}| = \gamma^{h,i}_{j,k}$. The 1-homogeneous property was introduced in [20] and has recently received a great deal of attention [11,15,17–19]. In this paper we give an algebraic description of the 1-homogeneous distance-regular graphs. Our results are similar to those appearing in [8] concerning the tight distance-regular graphs, similar to those appearing in [7] concerning the 2-homogeneous and almost 2-homogeneous bipartite distance-regular graphs, and similar to those in [9] for general graphs. 2-Homogeneous bipartite distance-regular graphs have been studied in [5,8,7,10,21–24,28].

We first consider a relaxation of 1-homogeneity. Fix a vertex x of Γ. Then Γ is said to be 1-*homogeneous with respect to* x whenever for all integers h, i, j, k, $(0 \leqslant h, i, j, k \leqslant d)$ there is a structure constant $\gamma^{h,i}_{j,k}(x)$ such that for all vertices y and z of Γ with $\partial(x, y) = h$, $\partial(y, z) = i$, $\partial(z, x) = 1$, the number $|\{w \in X \mid \partial(x, w) = j, \ \partial(y, w) = 1, \ \partial(z, w) = k\}| = \gamma^{h,i}_{j,k}(x)$. The following result links the combinatorial and algebraic structures of a 1-homogeneous distance-regular graph:

Theorem 1.1. *Let Γ denote a distance-regular graph with diameter d. Fix a base vertex x of Γ, and write $E^*_i = E^*_i(x)$ $(0 \leqslant i \leqslant d)$, $\mathcal{M}^* = \mathcal{M}^*(x)$, and $\mathcal{T} = \mathcal{T}(x)$. The following are equivalent.*

(i) *Γ is* 1*-homogeneous with respect to x.*
(ii) $\mathcal{T} E^*_1 = \mathcal{M}^* \mathcal{M} E^*_1$.
(iii) $\dim E^*_i \mathcal{T} E^*_1 \leqslant 3$ $(1 \leqslant i \leqslant d-1)$ *and* $\dim E^*_d \mathcal{T} E^*_1 \leqslant 2$.

We then study the decomposition of $\mathcal{T}E_1^*$ into minimal left ideals of $\mathcal{T}$ for 1-homogeneous distance-regular graphs. To describe this result precisely we recall some more facts and notation (see Section 7). Let $\mathcal{L}$ denote a minimal left ideal of $\mathcal{T}$. The *endpoint* and *diameter* of $\mathcal{L}$ are respectively the least i such that $E_i^*\mathcal{L} \neq 0$ and the number $|\{i \mid E_i^*\mathcal{L} \neq 0\}| - 1$. $\mathcal{L}$ is said to be *thin* if $\dim E_i^*\mathcal{L} \leqslant 1$ $(0 \leqslant i \leqslant d)$. Every left ideal of $\mathcal{T}$ decomposes into a direct sum of minimal left ideals. There is a unique minimal left ideal of $\mathcal{T}$ with endpoint 0 contained in $\mathcal{T}E_1^*$: It is thin, has diameter d, and is generated by JE_1^*. All other minimal left ideals of $\mathcal{T}$ contained in $\mathcal{T}E_1^*$ have endpoint 1 and diameter $d-1$ or $d-2$ but they need not be thin. We are ready to state our results concerning the decomposition of $\mathcal{T}E_1^*$ for the 1-homogeneous distance-regular graphs. The decomposition of $\mathcal{T}E_1^*$ is determined by which of the intersection numbers $a_i := p_{i1}^i$ are nonzero as follows.

Theorem 1.2. *Let Γ denote a distance-regular graph with diameter $d \geqslant 2$. Fix a base vertex x of Γ, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant d)$ and $\mathcal{T} = \mathcal{T}(x)$.*

(i) *Assume $a_i = 0$ $(0 \leqslant i \leqslant d)$. Then $\mathcal{T}E_1^* = \mathcal{T}JE_1^* \oplus \mathcal{L}$, where $\mathcal{L}$ is a thin minimal left ideal of $\mathcal{T}$ with endpoint 1 and diameter $d-2$. Moreover, Γ is 1-homogeneous.*

(ii) *Assume $a_i = 0$ $(0 \leqslant i \leqslant d)$. Then $\mathcal{T}E_1^* = \mathcal{T}JE_1^* \oplus \mathcal{L}$, where $\mathcal{L}$ is a thin minimal left ideal of $\mathcal{T}$ with endpoint 1 and diameter $d-1$. Moreover, Γ is 1-homogeneous.*

(iii) *Assume $a_1 = 0$, $a_d = 0$, $\exists i$ $(2 \leqslant i \leqslant d-1)$ $a_i \neq 0$. Then Γ is 1-homogeneous with respect to x if and only if $\mathcal{T}E_1^* = \mathcal{T}JE_1^* \oplus \mathcal{L}$, where $\mathcal{L}$ is a minimal left ideal of $\mathcal{T}$ with endpoint 1, and $\dim E_i^*\mathcal{L} = 1$ if $1 \leqslant i < i_1$, $\dim E_i^*\mathcal{L} = 2$ if $i_1 \leqslant i \leqslant i_2$, $\dim E_i^*\mathcal{L} = 1$ if $i_2 < i < d$, and $\dim E_i^*\mathcal{L} = 0$ if $i = d$, where $i_1 = \min\{i | a_i \neq 0\}$, $i_2 = \max\{i | a_i \neq 0\}$.*

(iv) *Assume $a_1 = 0$, $a_d \neq 0$, $\exists i$ $(2 \leqslant i \leqslant d-2)$ $a_i \neq 0$. Then Γ is 1-homogeneous with respect to x if and only if $\mathcal{T}E_1^* = \mathcal{T}JE_1^* \oplus \mathcal{L}$, where $\mathcal{L}$ is a minimal left ideal of $\mathcal{T}$ with endpoint 1, and $\dim E_i^*\mathcal{L} = 1$ if $1 \leqslant i < i_1$, $\dim E_i^*\mathcal{L} = 2$ if $i_1 \leqslant i \leqslant d-1$, and $\dim E_i^*\mathcal{L} = 1$ if $i = d$, where $i_1 = \min\{i | a_i \neq 0\}$, $i_2 = \max\{i | a_i \neq 0\}$.*

(v) *Assume $a_1 \neq 0$, $a_d = 0$. Then Γ is 1-homogeneous with respect to x if and only if $\mathcal{T}E_1^* = \mathcal{T}JE_1^* \oplus \mathcal{L}_1 \oplus \mathcal{L}_2$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are both thin minimal left ideals of $\mathcal{T}$ with endpoint 1 and diameter $d-2$.*

(vi) *Assume $a_1 \neq 0$, $a_d \neq 0$. Then Γ is 1-homogeneous with respect to x if and only if $\mathcal{T}E_1^* = \mathcal{T}JE_1^* \oplus \mathcal{L}_1 \oplus \mathcal{L}_2$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are both thin minimal left ideals of $\mathcal{T}$ with endpoint 1 and one has diameter $d-2$ and the other diameter $d-1$.*

Parts (i)–(vi) of Theorem 1.2 are interesting from an algebraic perspective. A distance-regular graph is said to be 1-*thin with respect to x* when every minimal left ideal of the Terwilliger algebra $\mathcal{T}(x)$ with endpoint one is thin. The 1-thin distance-regular graphs have recently received some attention [25,26,27]. Some analogous results for bipartite graphs (those which are "2-thin") have been treated in [6,7]. It is natural to consider the 1-thin distance-regular graphs with few minimal left ideals of endpoint one. Theorem 1.2 implies the following known result.

Theorem 1.3 (Collinis *[4]*, Curtin *[6]*). *Let Γ denote a distance-regular graph with diameter $d \geqslant 2$. Fix a base vertex x of Γ, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant d)$ and $\mathcal{T} = \mathcal{T}(x)$. Then*

$\mathcal{T}E_1^* = \mathcal{T}JE_1^* \oplus \mathcal{L}$, where $\mathcal{L}$ is a thin minimal left ideal of $\mathcal{T}$ with endpoint 1 *if and only if Γ is bipartite or almost bipartite.*

Theorem 1.2 “almost” describes the distance-regular graphs for which $\mathcal{T}E_1^* = \mathcal{T}JE_1^* \oplus \mathcal{L}_1 \oplus \mathcal{L}_2$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are both thin minimal left ideals of $\mathcal{T}$ with endpoint 1. The remaining case (where $\mathcal{L}_1$ and $\mathcal{L}_2$ both have diameter $d-1$) turns out to have a combinatorial characterization similar to the 1-homogeneous property.

Γ is said to be *pseudo*-1-*homogeneous* whenever for all integers h, i, j, k, $(0 \leqslant h, i, j, k \leqslant d$, but not $h = i = d)$ there are structure constants $\gamma_{j,k}^{h,i}$ such that for all vertices x, y, z of Γ with $\partial(x, y) = h$, $\partial(y, z) = i$, $\partial(z, x) = 1$, the number $|\{w \in X \mid \partial(x, w) = j,\ \partial(y, w) = 1,\ \partial(z, w) = k\}| = \gamma_{j,k}^{h,i}$. Observe that every 1-homogeneous distance-regular graph is pseudo-1-homogeneous. As in the 1-homogeneous case, we consider a relaxation of pseudo-1-homogeneity. Γ is said to be *pseudo*-1-*homogeneous with respect to x* whenever for all integers h, i, j, k, $(0 \leqslant h \leqslant d-1,\ 0 \leqslant i,\ j,\ k \leqslant d)$ there are structure constants $\gamma_{j,k}^{h,i}(x)$ such that for all vertices y, z of Γ with $\partial(x, y) = h$, $\partial(y, z) = i$, $\partial(z, x) = 1$, the number $|\{w \in X \mid \partial(x, w) = j,\ \partial(y, w) = 1,\ \partial(z, w) = k\}| = \gamma_{j,k}^{h,i}(x)$. (Although there are no structure constants $\gamma_{j,k}^{d,d-1}(x)$, pseudo-1-homogeneous with respect to every vertex implies pseudo-1-homogeneous). This property has the following $\mathcal{T}$-algebraic characterizations (in the case $a_1 \neq 0$). First there is an analog of Theorem 1.1:

Theorem 1.4. *Let Γ denote a distance-regular graph with diameter $d \geqslant 2$. Fix a base vertex x of Γ, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant d)$ and $\mathcal{T} = \mathcal{T}(x)$. Assume that $a_1 \neq 0$. Then the following are equivalent.*

(i) *Γ is pseudo-1-homogeneous with respect to x.*
(ii) *$\mathcal{T}E_1^*$ is spanned by $\{E_0^*AE_1^*\} \cup \{E_i^*A_{i-1}E_1^*, E_i^*A_iE_1^*, E_i^*A_{i+1}E_1^* \mid 1 \leqslant i \leqslant d-1\} \cup \{E_d^*A_{d-1}E_1^*, E_d^*A_dE_1^*, E_d^*AE_d^*A_{d-1}E_1^*\}$.*
(iii) dim $E_i^*\mathcal{T}E_1^* \leqslant 3$ $(1 \leqslant i \leqslant d)$.

With this key result, we may describe the decomposition of $\mathcal{T}E_1^*$.

Theorem 1.5. *Let Γ denote a distance-regular graph with diameter $d \geqslant 2$. Fix a base vertex x of Γ, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant d)$ and $\mathcal{T} = \mathcal{T}(x)$. Then the following are equivalent.*

(i) *Γ is pseudo-1-homogeneous with respect to x with $a_1 \neq 0$ but not 1-homogeneous with respect to x.*
(ii) *$\mathcal{T}E_1^* = \mathcal{T}JE_1^* \oplus \mathcal{L}_1 \oplus \mathcal{L}_2$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are both thin minimal left ideals of $\mathcal{T}$ with endpoint* 1 *and diameter $d-1$.*

This gives the following result concerning 1-thin distance-regular graphs.

Theorem 1.6. *Let Γ denote a distance-regular graph with diameter $d \geqslant 2$. Fix a base vertex x of Γ, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant d)$ and $\mathcal{T} = \mathcal{T}(x)$. Then the following are equivalent.*

(i) *Γ is pseudo-1-homogeneous with respect to x with $a_1 \neq 0$.*
(ii) *$\mathcal{T}E_1^* = \mathcal{T}JE_1^* \oplus \mathcal{L}_1 \oplus \mathcal{L}_2$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are both thin minimal left ideals of $\mathcal{T}$ with endpoint* 1.

We shall prove Theorem 1.1 in Section 5, Theorem 1.4 in Section 6, and Theorems 1.2, 1.3, 1.5, and 1.6 in Section 8. We now begin with some background and then develop the results used in these proofs.

2. Background

We recall some basic facts concerning distance-regular graphs and their Bose–Mesner algebras (see [1–3]). Let Γ denote a distance-regular graph with vertex set X and diameter d. Let ∂ denote the shortest-path distance-function on Γ. For all integers i and for all vertices x of Γ, let $\Gamma_i(x) = \{y \in X \mid \partial(x, y) = i\}$. With this notation $p^h_{ij} = |\Gamma_i(x) \cap \Gamma_j(y)|$ for any vertices x, y of Γ with $\partial(x, y) = h$. Clearly $p^h_{ij} = 0$ if and only if $p^i_{jh} = 0$ if and only if $p^j_{hi} = 0$ because these numbers count vertices in related configurations. The triangle inequality for ∂ implies that p^h_{ij} is zero if one of i, j, k is greater than the sum of the other two and that p^h_{ij} is nonzero if one of i, j, k is equal to the sum of the other two. Abbreviate $c_i = p^i_{i-11}$ $(1 \leqslant i \leqslant d)$, $a_i = p^i_{i1}$ $(0 \leqslant i \leqslant d)$, and $b_i = p^i_{i+11}$ $(0 \leqslant i \leqslant d-1)$. We set $c_0 = b_d = 0$ and $c_i = a_i = b_i = 0$ for $i < 0$ or $i > d$. Observe that Γ is regular with valency b_0, and that

$$b_0 = c_i + a_i + b_i \qquad (0 \leqslant i \leqslant d). \tag{1}$$

Let $Mat_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra of complex matrices whose rows and columns are indexed by X. For $M \in Mat_X(\mathbb{C})$ and for $a, b \in X$, let $M(a, b)$ denote the (a, b)-entry of M. For all integers i, define $A_i \in Mat_X(\mathbb{C})$ to be the matrix with (x, y)-entry

$$A_i(x, y) = \begin{cases} 1 & \partial(x, y) = i, \\ 0 & \text{otherwise} \end{cases} \qquad (x, y \in X).$$

The matrix A_i is called the ith *distance-matrix* of Γ. Observe that $A_0 = I$ (the identity matrix), that $A := A_1$ is the usual *adjacency matrix*, and that $A_i = 0$ if $i < 0$ or $i > d$. Since Γ is connected,

$$\sum_{i=0}^{d} A_i = J, \tag{2}$$

where J denotes the all-ones matrix of $Mat_X(\mathbb{C})$. Furthermore,

$$A_i A_j = A_j A_i = \sum_{h=0}^{d} p^h_{ij} A_h. \tag{3}$$

In particular,

$$A A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1} \qquad (1 \leqslant i \leqslant d-1). \tag{4}$$

Let $\mathcal{M}$ denote the linear span of $\{A_i \mid 0 \leqslant i \leqslant d\}$. Observe that $\mathcal{M}$ is a commutative subalgebra of $Mat_X(\mathbb{C})$. Also observe that $\mathcal{M}$ is generated by $A = A_1$ by (4). It turns out that this adjacency algebra is a Bose–Mesner algebra, so we refer to $\mathcal{M}$ as the *Bose–Mesner algebra* of Γ.

We now recall the Terwilliger algebra of a distance-regular graph (see [25]). Given a "base vertex" $x \in X$, for all integers i define $E_i^* = E_i^*(x) \in Mat_X(\mathbb{C})$ $(0 \leqslant i \leqslant d)$ to be the diagonal matrix with (y, y)-entry

$$E_i^*(y, y) = \begin{cases} 1 & \partial(x, y) = i, \\ 0 & \text{otherwise} \end{cases} \qquad (y \in X).$$

The matrix E_i^* is called the *ith dual idempotent with respect to x*. We shall always set $E_i^* = 0$ for $i < 0$ or $i > d$. Let $\mathcal{M}^* = \mathcal{M}^*(x)$ denote the subalgebra of $Mat_X(\mathbb{C})$ generated by E_0^*, $E_1^*, \ldots, E_d^*$. $\mathcal{M}^*$ is called the *dual Bose–Mesner algebra of Γ with respect to x*. The dual idempotents form a linear basis for $\mathcal{M}^*$ and satisfy the relations

$$\sum_{i=1}^{d} E_i^* = I, \tag{5}$$

$$E_i^* E_j^* = \delta_{ij} E_i^* \quad (0 \leqslant i, \ j \leqslant d). \tag{6}$$

Let $\mathcal{T} = \mathcal{T}(x)$ denote the matrix subalgebra of $Mat_X(\mathbb{C})$ generated by $\mathcal{M}$ and $\mathcal{M}^*(x)$. $\mathcal{T}$ is called the *Terwilliger* (or *subconstituent*) *algebra of Γ with respect to x*. Dual Bose–Mesner algebras and Terwilliger algebras were introduced in [25] to study distance-regular graphs and association schemes. The following generators of $\mathcal{T}$ are useful.

Lemma 2.1 (*Terwilliger [25]*). *Let Γ denote a distance-regular graph with diameter d. Fix any vertex x of Γ, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant d)$ and $\mathcal{T} = \mathcal{T}(x)$. Define $L = L(x)$, $F = F(x)$, $R = R(x)$ by*

$$L = \sum_{i=1}^{d} E_{i-1}^* A E_i^*, \qquad F = \sum_{i=0}^{d} E_i^* A E_i^*, \qquad R = \sum_{i=0}^{d-1} E_{i+1}^* A E_i^*. \tag{7}$$

Then

$$A = L + F + R. \tag{8}$$

In particular, L, F, R, $E_0^*, E_1^*, \ldots, E_d^*$ generate $\mathcal{T}$. Note $L^{\mathrm{t}} = R$ and $F^{\mathrm{t}} = F$.

3. Triples of vertices

Let Γ denote a distance-regular graph with vertex set X. The matrices of $Mat_X(\mathbb{C})$ naturally encode information concerning pairs of vertices. In this section we discuss how Terwilliger algebras encode information concerning certain triples of vertices.

Lemma 3.1. *Let Γ denote a distance-regular graph with diameter d. Fix a base vertex x of Γ, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant d)$. Then for all h, i, j $(0 \leqslant h, \ i, \ j \leqslant d)$ and* *for all vertices y, z of Γ, the (y,z)-entry of $E_h^* A_i E_j^*$ is*

$$E_h^* A_i E_j^*(y,z) = \begin{cases} 1 & \text{if } \partial(x,y)=h,\ \partial(y,z)=i,\ \text{and } \partial(z,x)=j, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Straight forward from the definitions of the dual idempotents, the distance-matrices, and matrix multiplication. □

Corollary 3.2. *Let Γ denote a distance-regular graph with diameter d. Fix a base vertex x of Γ, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant d)$. Then the set of all nonzero matrices of the form $E_i^* A_j E_k^*$ is linearly independent.*

Proof. These matrices have no common nonzero entries by Lemma 3.1. □

Corollary 3.3. *Let Γ denote a distance-regular graph with diameter d. Fix a base vertex x of Γ, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant d)$. For all i, j, k $(0 \leqslant i,\ j,\ k \leqslant d)$, $E_i^* A_j E_k^* = 0$ if and only if $p^i_{jk} = 0$. In particular, $E_i^* A_j E_k^*$ is zero if one of i, j, k is greater than the sum of the other two and $E_i^* A_j E_k^*$ is nonzero if one of i, j, k is equal to the sum of the other two.*

Proof. Observe that $p^i_{jk} \neq 0$ if and only if there exist $y, z \in X$ with $\partial(x,y) = i$, $\partial(y,z) = j$, $\partial(z,x) = k$ if and only if the (y,z)-entry of $E_i^* A_j E_k^*$ is 1. □

Let Γ denote a distance-regular graph with diameter d. For all vertices x, y, z of Γ, define $\gamma_{j,k}(x,y,z)$ for all integers j, k by

$$\gamma_{j,k}(x,y,z) := |\Gamma_j(x) \cap \Gamma_1(y) \cap \Gamma_k(z)|.$$

Fix a vertex x of Γ, and suppose that Γ is 1-homogeneous with respect to x (respectively pseudo-1-homogeneous with respect to x). Then there are integral structure constants $\gamma^{h,i}_{j,k}(x)$ $(0 \leqslant h,\ i,\ j,\ k \leqslant d)$ (respectively $0 \leqslant h \leqslant d-1,\ 0 \leqslant i,\ j,\ k \leqslant d$) such that $\gamma^{h,i}_{j,k}(x) = \gamma_{j,k}(x,y,z)$ for all vertices y, z of Γ with $\partial(x,y) = h$, $\partial(y,z) = i$, $\partial(z,x) = 1$. In either case, we set $\gamma^{h,i}_{j,k}(x) = 0$ when there are no vertices y, z such that $\partial(x,y) = h$, $\partial(y,z) = i$, $\partial(z,x) = 1$. We also set $\gamma^{h,i}_{j,k}(x) = 0$ when at least one of h, i, j, k is negative or greater than d. For given vertices x, y, z of Γ with $\partial(x,z) = 1$, distance-regularity implies that the numbers $\gamma_{j,k}(x,y,z)$ are not independent.

Lemma 3.4. *Let Γ denote a distance-regular graph with diameter d. Fix i $(0 \leqslant i \leqslant d)$.*
(i) *For all vertices x, y, z of Γ with $\partial(x,y) = i$, $\partial(y,z) = i+1$, $\partial(z,x) = 1$,*

$$\gamma_{i+1,i+2}(x,y,z) = b_{i+1}, \tag{9}$$
$$\gamma_{i+1,i+1}(x,y,z) + \gamma_{i,i+1}(x,y,z) = a_{i+1}, \tag{10}$$
$$\gamma_{i+1,i}(x,y,z) + \gamma_{i,i}(x,y,z) + \gamma_{i-1,i}(x,y,z) = c_{i+1}, \tag{11}$$
$$\gamma_{i+1,i+2}(x,y,z) + \gamma_{i+1,i+1}(x,y,z) + \gamma_{i+1,i}(x,y,z) = b_i, \tag{12}$$
$$\gamma_{i,i+1}(x,y,z) + \gamma_{i,i}(x,y,z) = a_i, \tag{13}$$
$$\gamma_{i-1,i}(x,y,z) = c_i. \tag{14}$$

(ii) *For all vertices x, y, z of Γ with $\partial(x,y)=i$, $\partial(y,z)=i$, $\partial(z,x)=1$,*

$$\gamma_{i,i+1}(x,y,z)+\gamma_{i+1,i+1}(x,y,z)=b_i, \tag{15}$$
$$\gamma_{i+1,i}(x,y,z)+\gamma_{i,i}(x,y,z)+\gamma_{i-1,i}(x,y,z)=a_i, \tag{16}$$
$$\gamma_{i,i-1}(x,y,z)+\gamma_{i-1,i-1}(x,y,z)=c_i. \tag{17}$$

Proof. Fix vertices x, y of Γ, and let $h=\partial(x,y)$. Then for all j $(0\leqslant j\leqslant d)$, the set $\Gamma_j(x)\cap\Gamma_1(y)$ has size p^h_{j1}. Now fix a vertex z adjacent to x. Then $\Gamma_j(x)\cap\Gamma_1(y)$ is a disjoint union of all sets $\Gamma_j(x)\cap\Gamma_1(y)\cap\Gamma_k(z)$ $(0\leqslant k\leqslant d)$. Observe that $|\Gamma_j(x)\cap\Gamma_1(y)\cap\Gamma_k(z)|=\gamma_{j,k}(x,y,z)$, so $p^h_{j1}=\sum_{k=0}^{d}\gamma_{j,k}(x,y,z)$. The result follows since $\gamma_{j,k}(x,y,z)=0$ unless $|h-j|\leqslant 1$, $|\partial(y,z)-k|\leqslant 1$, and $|j-k|\leqslant 1$ by the triangle inequality. □

Swapping x and z yields equations identical to (9)–(17) except that $\gamma_{j,k}(x,y,z)$ is replaced by $\gamma_{k,j}(z,y,x)$.

Lemma 3.5. *Let Γ denote a distance-regular graph with diameter d. Fix a base vertex x of Γ and write $E^*_i=E^*_i(x)$ $(0\leqslant i\leqslant d)$. Then for all h, j, k $(0\leqslant h,\ j,\ k\leqslant d)$ and for all vertices y, z of Γ, the (y,z)-entry of $E^*_hAE^*_jA_kE^*_1$ is*

$$E^*_hAE^*_jA_kE^*_1(y,z)=\begin{cases}\gamma_{j,k}(x,y,z) & \textit{if } \partial(x,y)=h,\ \partial(z,x)=1,\\ 0 & \textit{otherwise}.\end{cases}$$

Proof. Parse this product as $E^*_hAE^*_j\times E^*_jA_kE^*_1$ using (6). By the definition of matrix multiplication, this product has (y,z)-entry $\sum_w E^*_hAE^*_j(y,w)E^*_jA_kE^*_1(w,z)$. By Lemma 3.1, the first factor is 1 if $\partial(x,y)=h$, $\partial(y,w)=1$, and $\partial(w,x)=j$ and zero otherwise. The second factor is 1 if $\partial(w,x)=j$, $\partial(w,z)=k$, and $\partial(x,z)=1$ and zero otherwise. Thus the summand is 0 unless $\partial(x,y)=h$ and $\partial(z,x)=1$. If $\partial(x,y)=h$ and $\partial(z,x)=1$, then the summand is 1 when $w\in\Gamma_j(x)\cap\Gamma_1(y)\cap\Gamma_k(z)$ and 0 otherwise. □

Corollary 3.6. *Let Γ denote a distance-regular graph with diameter d. Fix a base vertex x of Γ, and write $E^*_i=E^*_i(x)$ $(0\leqslant i\leqslant d)$.*

(i) *Fix i $(1\leqslant i\leqslant d)$. For all vertices y, z of Γ with $\partial(x,y)=i-1$, $\partial(x,z)=1$.*

$$LE^*_iA_{i-1}E^*_1(y,z)=\gamma_{i,i-1}(x,y,z), \tag{18}$$
$$LE^*_iA_iE^*_1(y,z)=\gamma_{i,i}(x,y,z), \tag{19}$$
$$LE^*_iA_{i+1}E^*_1(y,z)=\gamma_{i,i+1}(x,y,z). \tag{20}$$

Furthermore, both sides of (18)–(20) *are zero unless $\partial(y,z)$ lies respectively in $\{i-2,i-1,i\}$, $\{i-1,i\}$, and $\{i\}$.*

(ii) *Fix i $(0\leqslant i\leqslant d)$. For all vertices y, z of Γ with $\partial(x,y)=i$, $\partial(x,z)=1$.*

$$FE^*_iA_{i-1}E^*_1(y,z)=\gamma_{i,i-1}(x,y,z), \tag{21}$$
$$FE^*_iA_iE^*_1(y,z)=\gamma_{i,i}(x,y,z), \tag{22}$$
$$FE^*_iA_{i+1}E^*_1(y,z)=\gamma_{i,i+1}(x,y,z). \tag{23}$$

Furthermore, both sides of (21)–(23) *are zero unless* $\partial(y,z)$ *lies respectively in* $\{i-1, i\}$, $\{i-1, i, i+1\}$, *and* $\{i, i+1\}$.

(iii) *Fix* i $(0 \leqslant i \leqslant d-1)$. *For all vertices* y, z *of* Γ *with* $\partial(x,y) = i+1$, $\partial(x,z) = 1$.

$$RE_i^* A_{i-1} E_1^*(y,z) = \gamma_{i,i-1}(x,y,z), \tag{24}$$

$$RE_i^* A_i E_1^*(y,z) = \gamma_{i,i,i}(x,y,z), \tag{25}$$

$$RE_i^* A_{i+1} E_1^*(y,z) = \gamma_{i,i+1}(x,y,z). \tag{26}$$

Furthermore, both sides of (24)–(26) *are zero unless* $\partial(y,z)$ *lies respectively in* $\{i\}$, $\{i, i+1\}$, *and* $\{i, i+1, i+2\}$.

Proof. Eqs. (18)–(26) are special cases of Lemma 3.5 after expanding L, F, R with (7) and simplifying with (6). The restrictions on $\partial(y,z)$ follow from Corollary 3.3 and the triangle inequality. □

4. The space $\mathcal{T} E_1^*$

In this section we consider the action of a Terwilliger algebra $\mathcal{T}$ of a distance-regular graph on the subspace $\mathcal{M}^*\mathcal{M}E_1^* \subseteq \mathcal{T}E_1^*$ in order to understand the case of equality.

Lemma 4.1. *Let* Γ *denote a distance-regular graph with diameter* d. *Fix a base vertex* x *of* Γ, *and write* $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant d)$, $\mathcal{M}^* = \mathcal{M}^*(x)$. *Then* $\mathcal{M}^*\mathcal{M}E_1^*$ *is the linear span of* $\{E_0^* A E_1^*\} \cup \{E_i^* A_{i-1} E_1^*, E_i^* A_i E_1^*, E_i^* A_{i+1} E_1^* \mid 1 \leqslant i \leqslant d-1\} \cup \{E_d^* A_{d-1} E_1^*, E_d^* A_d E_1^*\}$.

Proof. By construction $\mathcal{M}^*\mathcal{M}E_1^*$ is spanned by all matrices of the form $E_i^* A_j E_1^*$. But by Corollary 3.3, only those in the statement of the lemma may be nonzero. □

Lemma 4.2. *Let* Γ *denote a distance-regular graph with diameter* d. *Fix a base vertex* x *of* Γ, *and write* $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant d)$. *Then*

$$E_i^* J E_1^* = E_i^* A_{i-1} E_1^* + E_i^* A_i E_1^* + E_i^* A_{i+1} E_1^* \qquad (0 \leqslant i \leqslant d). \tag{27}$$

Proof. Clear from (2) and Corollary 3.3. □

Lemma 4.3 (*Terwilliger [25]*). *Let* Γ *denote a distance-regular graph with diameter* d. *Fix a base vertex* x *of* Γ, *and write* $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant d)$. *Then*

$$LE_i^* J = b_{i-1} E_{i-1}^* J \qquad (1 \leqslant i \leqslant d), \tag{28}$$

$$FE_i^* J = a_i E_i^* J \qquad (0 \leqslant i \leqslant d), \tag{29}$$

$$RE_i^* J = c_{i+1} E_{i+1}^* J \qquad (0 \leqslant i \leqslant d-1). \tag{30}$$

Proof. For all vertices y, z of Γ, the (y, z)-entry of $AE_i^* J$ is

$$AE_i^* J(y, z) = \sum_w AE_i^*(y, w)J(w, z) = \sum_{w \in \Gamma_i(x)} A(y, w) = |\Gamma_i(x) \cap \Gamma_1(y)| = p_{i1}^h,$$

where $h = \partial(x, y)$. The (y, z)-entry of $E_h^* J$ is 1 if $\partial(x, y) = h$ and 0 otherwise. Thus $AE_i^* J = b_{i-1}E_{i-1}^* J + a_i E_i^* J + c_{i+1}E_{i+1}^* J$. Multiplying this equation on the left by E_{i-1}^*, E_i^*, and E_{i+1}^* and simplifying with (6) and (7) gives (28)–(30). $\square$

Lemma 4.4. *Let Γ denote a distance-regular graph with diameter d. Fix a base vertex x of Γ, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant d)$. Then*

$$LE_i^* A_{i+1} E_1^* = b_i E_{i-1}^* A_i E_1^* \qquad (1 \leqslant i \leqslant d), \tag{31}$$

$$RE_i^* A_{i-1} E_1^* = c_i E_{i+1}^* A_i E_1^* \qquad (0 \leqslant i \leqslant d-1). \tag{32}$$

Proof. Eqs. (31) and (32) follow from (20), (24), (9), and (14). $\square$

An immediate consequence of (32) is that

$$E_i^* A_{i-1} E_1^* = (c_1 c_2 \cdots c_{i-1})^{-1} E_i^* R^{i-1} E_1^* \qquad (1 \leqslant i \leqslant d). \tag{33}$$

Lemma 4.5. *Let Γ denote a distance-regular graph with diameter d. Fix a base vertex x of Γ, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant d)$. Then for all i $(0 \leqslant i \leqslant d)$*

$$LE_{i+1}^* A_i E_1^* - FE_i^* A_{i+1} E_1^* = (c_{i+1} + b_i - b_0)E_i^* A_{i+1} E_1^* + b_i E_i^* A_{i-1} E_1^*, \tag{34}$$

$$RE_{i-1}^* A_i E_1^* - FE_i^* A_{i-1} E_1^* = (b_{i-1} + c_i - b_0)E_i^* A_{i-1} E_1^* + c_i E_i^* A_{i+1} E_1^*. \tag{35}$$

In particular, if $a_i = 0$ for some i $(0 \leqslant i \leqslant d)$, then

$$LE_{i+1}^* A_i E_1^* = (c_{i+1} - c_i)E_i^* A_{i+1} E_1^* + b_i E_i^* A_{i-1} E_1^*, \tag{36}$$

$$LE_{i+1}^* A_{i+1} E_1^* = a_{i+1} E_i^* A_{i+1} E_1^*, \tag{37}$$

$$RE_{i-1}^* A_i E_1^* = (b_{i-1} - b_i)E_i^* A_{i-1} E_1^* + c_i E_i^* A_{i+1} E_1^*, \tag{38}$$

$$RE_{i-1}^* A_{i-1} E_1^* = a_{i-1} E_i^* A_{i-1} E_1^*. \tag{39}$$

Proof. Eq. (34) is verified entry-by-entry. Both sides have (y, z)-entry equal to zero unless $\partial(x, y) = i$, $\partial(z, x) = 1$, and $\partial(y, z) \in \{i-1, i, i+1\}$, so fix vertices y, z of Γ with $\partial(x, y) = i$, $\partial(z, x) = 1$. First suppose $\partial(y, z) = i - 1$. Then $LE_{i+1}^* A_i E_1^*(y, z) = \gamma_{i+1,i}(x, y, z)$ and $FE_i^* A_{i+1} E_1^*(y, z) = 0$ by (18) and Corollary 3.6. But $\gamma_{i+1,i}(x, y, z) = b_i$ by (9), so the left-hand side of (34) has (y, z)-entry b_i. By Lemma 3.1, the right-hand side of (34) also has (y, z)-entry b_i. Next, suppose $\partial(y, z) = i$. Then $LE_{i+1}^* A_i E_1^*(y, z) = \gamma_{i+1,i}(x, y, z)$ and $FE_i^* A_{i+1} E_1^*(y, z) = \gamma_{i,i+1}(x, y, z)$. But both of these terms are equal to $b_i - \gamma_{i+1,i+1}(x, y, z)$ by (15), so the left-hand side of (34) has (y, z)-entry zero. The right-hand side clearly has (y, z)-entry zero. Finally, suppose $\partial(y, z) = i + 1$. Then $LE_{i+1}^* A_i E_1^*(y, z) = \gamma_{i+1,i}(x, y, z)$ and $FE_i^* A_{i+1} E_1^*(y, z) = \gamma_{i,i+1}(x, y, z)$. But $\gamma_{i+1,i}$

$(x, y, z) = b_i - \gamma_{i+1,i+2}(x, y, z) - \gamma_{i+1,i+1}(x, y, z) = b_i - b_{i+1} - a_{i+1} + \gamma_{i,i+1}(x, y, z)$ by (9)–(12). Thus the left-hand side (34) has (y, z)-entry $c_{i+1} - b_0 + b_i$ by (1). The right-hand side clearly has (y, z)-entry $c_{i+1} - b_0 + b_i$ too. Thus (34) holds. A similar argument verifies (35).

To see (36) and (38), note that $a_i = 0$ implies that $FE_i^* A_{i+1} E_1^* = 0$ and $E_i^* A_i E_1^* = 0$ by Corollary 3.3, and it implies that $b_0 - b_i = c_i$ by (1). To see (37) and (39), observe that $E_j^* A_j E_1^* = E_j^* J E_1^* - E_j^* A_{j-1} E_1^* - E_j^* A_{j+1} E_1^*$ by (27), that the action of L on the right-hand side is given by (28), (31), (36), and that the action of R on the right-hand side is given by (30), (32), (38). $\square$

Lemma 4.6. *Let Γ denote a distance-regular graph with diameter d. Fix a base vertex x of Γ, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant d)$. Suppose $a_i = 0$ for some i $(0 \leqslant i \leqslant d)$. Then*

$$\begin{aligned} LE_i^* A_{i-1} E_1^* &= b_{i-1} E_{i-1}^* A_{i-2} E_1^* + b_{i-1} E_{i-1}^* A_{i-1} E_1^* \\ &\quad + (b_{i-1} - b_i) E_{i-1}^* A_i E_1^*, \end{aligned} \tag{40}$$

$$\begin{aligned} RE_i^* A_{i+1} E_1^* &= c_{i+1} E_{i+1}^* A_{i+2} E_1^* + c_{i+1} E_{i+1}^* A_{i+1} E_1^* \\ &\quad + (c_{i+1} - c_i) E_{i+1}^* A_i E_1^*. \end{aligned} \tag{41}$$

Proof. Apply arguments similar to those used in the proof of Lemma 4.5. $\square$

5. 1-Homogeneous distance-regular graphs

In this section we characterize the 1-homogeneous property for distance-regular graphs in terms of the left ideal $\mathcal{T}E_1^*$ of the Terwilliger algebra by proving Theorem 1.1. We then discuss some consequences of this result.

Proof of Theorem 1.1. (i)⇒(ii): Since $E_1^* \in \mathcal{M}^*\mathcal{M}E_1^* \subseteq \mathcal{T}E_1^*$, it is enough to show that $\mathcal{M}^*\mathcal{M}E_1^*$ is closed under left multiplication by the generators $L, F, R, E_0^*, E_1^*, \ldots, E_d^*$ of $\mathcal{T}$ from Lemma 2.1. It is closed under left multiplication by the E_i^* by (6) and Lemma 4.1. Observe that for vertices y, z of Γ with $\partial(x, y) = h$, $\partial(y, z) = i$, $\partial(z, x) = 1$, the entry $E_h^* A E_j^* A_k E_1^*(y, z) = \gamma_{j,k}(x, y, z) = \gamma_{j,k}^{h,i}(x)$ by Lemma 3.5 and 1-homogeneity with respect to x. Thus Lemmas 3.1 and 3.5 imply that $E_h^* A E_j^* A_k E_1^* = \sum_{i=0}^{d} \gamma_{j,k}^{h,i}(x) E_h^* A_i E_1^*$. It follows that $\mathcal{M}^*\mathcal{M}E_1^*$ is closed under left multiplication by L, F, and R. Thus (ii) holds.

(ii)⇒(i): Observe that $E_h^* A E_j^* A_k E_1^* \in \mathcal{T}E_1^*$, so $E_h^* A E_j^* A_k E_1^* = \sum_{n=0}^{d} \alpha_{j,k}^{h,n} E_h^* A_n E_1^*$ for some scalars $\alpha_{j,k}^{h,n} = \alpha_{j,k}^{h,n}(x)$ by the assumption that $\mathcal{T}E_1^* = \mathcal{M}^*\mathcal{M}E_1^*$ and Lemma 4.1. Now for all vertices y, z of Γ with $\partial(x, y) = h$, $\partial(y, z) = i$, and $\partial(z, x) = 1$, the (y, z)-entry of $E_h^* A E_j^* A_k E_1^*$ is $\gamma_{j,k}(x, y, z)$ by Lemma 3.5. But by Lemma 3.1, the (y, z)-entry of $\sum_{n=0}^{d} \alpha_{j,k}^{h,n} E_h^* A_n E_1^*$ is $\alpha_{j,k}^{h,i}$. Thus $\gamma_{j,k}(x, y, z) = \alpha_{j,k}^{h,i}(x)$. Hence Γ is 1-homogeneous with respect to x.

(ii)⇒(iii): Clear from Lemma 4.1.

(iii)⇒(ii): As above, it is enough to show that $\mathcal{M}^*\mathcal{M}E_1^*$ is closed under left multiplication by L, F, R. First, suppose $a_i = 0$ for some i $(0 \leqslant i \leqslant d)$. Then $FE_i^* A_{i+1} E_1^* =$

0, $FE_i^*A_{i-1}E_1^* = 0$, and $E_i^*A_iE_1^* = 0$ by Corollary 3.3. In addition, $LE_i^*A_{i-1}E_1^*$, $LE_i^*A_{i+1}E_1^*$, $RE_i^*A_{i+1}E_1^*$, $RE_i^*A_{i-1}E_1^* \in \mathcal{M}^*\mathcal{M}E_1^*$ by (31), (32), (40), and (41).

Now suppose $a_i \neq 0$ for some i $(1 \leqslant i \leqslant d)$. If $i < d-1$, then $E_i^*\mathcal{T}E_1^*$ has linear basis $\{E_i^*A_{i-1}E_1^*, E_i^*A_iE_1^*, E_i^*A_{i+1}E_1^*\}$ by Corollaries 3.2 and 3.3 and since $\dim E_i^*\mathcal{T}E_1^* \leqslant 3$ by assumption. Similarly, if $i = d$, then $E_d^*\mathcal{T}E_1^*$ has linear basis $\{E_d^*A_{d-1}E_1^*, E_d^*A_dE_1^*\}$. Thus it must be the case that $FE_i^*A_{i-1}E_1^*$, $FE_i^*A_iE_1^*$, and $FE_i^*A_{i+1}E_1^* \in \mathcal{M}^*\mathcal{M}E_1^*$. If $a_{i-1} = 0$, then $LE_i^*A_{i-1}E_1^*$, $LE_i^*A_iE_1^*$, $LE_i^*A_{i+1}E_1^* \in \mathcal{M}^*\mathcal{M}E_1^*$ by (36), (37), and (31). If $a_{i-1} \neq 0$, then $E_{i-1}^*\mathcal{T}E_1^* = \operatorname{span}\left\{E_{i-1}^*A_{i-2}E_1^*, E_{i-1}^*A_{i-1}E_1^*, E_{i-1}^*A_iE_1^*\right\}$, so it must be the case that $LE_i^*A_{i-1}E_1^*$, $LE_i^*A_iE_1^*$, $LE_i^*A_{i+1}E_1^* \in \mathcal{M}^*\mathcal{M}E_1^*$. A similar argument based on a_{i+1} shows that $RE_i^*A_{i-1}E_1^*$, $RE_i^*A_iE_1^*$, $RE_i^*A_{i+1}E_1^* \in \mathcal{M}^*\mathcal{M}E_1^*$. □

Corollary 5.1. *Let Γ denote a distance-regular graph with diameter d. Fix a base vertex x of Γ, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant d)$ and $\mathcal{T} = \mathcal{T}(x)$. Suppose Γ is 1-homogeneous with respect to x. Then the following hold.*

- (i) *$E_0^*\mathcal{T}E_1^*$ has linear basis $\{E_0^*AE_1^*\}$.*
- (ii) *For $1 \leqslant i \leqslant d-1$, $E_i^*\mathcal{T}E_1^*$ has linear basis $\{E_i^*A_{i-1}E_1^*, E_i^*A_iE_1^*, E_i^*A_{i+1}E_1^*\}$ if $a_i \neq 0$ and it has linear basis $\{E_i^*A_{i-1}E_1^*, E_i^*A_{i+1}E_1^*\}$ if $a_i = 0$.*
- (iii) *$E_d^*\mathcal{T}E_1^*$ has linear basis $\{E_d^*A_{d-1}E_1^*, E_d^*A_dE_1^*\}$ if $a_d \neq 0$, and it has linear basis $\{E_d^*A_{d-1}E_1^*\}$ if $a_d = 0$.*

Proof. Clear from Corollary 3.3, Lemma 4.1, and Theorem 1.1. □

Since the nonzero a_i play an important role in determining the basis for $\mathcal{T}E_1^*$ in the 1-homogeneous case, we recall a few facts about these intersection numbers.

Lemma 5.2. *Let Γ denote a distance-regular graph with diameter d.*

- (i) [3, Proposition 5.5.1(i)]. *If $a_i = 0$ for some i $(1 \leqslant i \leqslant d-1)$, then $a_1 = 0$.*
- (ii) [3, Proposition 5.5.7] *If $a_i \neq 0$ for some i $(1 \leqslant i \leqslant d)$, then there are numbers i_1 and i_2 such that $a_\ell \neq 0$ if and only if $i_1 \leqslant \ell \leqslant i_2$. Moreover, $i_1 + i_2 \geqslant d$.*

It is well-known that a distance-regular graph is *bipartite* if and only if $a_0 = a_1 = \cdots = a_d = 0$. By analogy, a distance-regular graph is called *almost bipartite* when $a_0 = a_1 = \cdots = a_{d-1} = 0$ but $a_d \neq 0$.

Lemma 5.3. *Let Γ denote a distance-regular graph. If Γ is bipartite or almost bipartite, then Γ is 1-homogeneous.*

Proof. Observe that there are no triples of vertices x, y, z of Γ with $\partial(x,z) = 1$, $\partial(x,y) = h$, and $\partial(y,z) = j$ unless $|h-j| = 1$ or $h = i = d$ (the latter only in the almost bipartite case). By possibly reversing the roles of x and y, we may assume that either $\partial(x,y) = i$ and $\partial(y,z) = i-1$ for some i $(1 \leqslant i \leqslant d)$ or $\partial(x,y) = \partial(y,z) = d$ and Γ is almost bipartite. In the former case, all neighbors of y are counted by $\gamma_{i-1,i-2}(x,y,z) = c_{i-1}$, $\gamma_{i-1,i}(x,y,z) = c_i - c_{i-1}$, and $\gamma_{i+1,i}(x,y,z) = b_i$. In the latter case, all neighbors of y are counted by $\gamma_{d,d-1}(x,y,z) = \gamma_{d-1,d}(x,y,z) = c_d$ and $\gamma_{d,d}(x,y,z) = a_d - c_d$. Thus Γ is 1-homogeneous. □

6. Pseudo-1-homogeneous distance-regular graphs

In this section we prove Theorem 1.4 which characterizes the pseudo-1-homogeneous property in terms of the space $\mathcal{T}E_1^*$. Although the result is similar to Theorem 1.1, its proof is much more involved. Moreover, Theorem 1.4 only applies to the case $a_1 \neq 0$. We begin with some preliminary lemmas.

Lemma 6.1. *Let Γ denote a distance-regular graph with diameter $d \geqslant 2$. Fix a base vertex x of Γ, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant d)$. Suppose that Γ is pseudo-1-homogeneous with respect to x, and write $\gamma_{j,k}^{h,i} = \gamma_{j,k}^{h,i}(x)$ $(0 \leqslant h \leqslant d-1,\ 0 \leqslant i,\ j,\ k \leqslant d)$. Then*

$$RE_i^* A_i E_1^* = \gamma_{i,i}^{i+1,i} E_{i+1}^* A_i E_1^* + \gamma_{i,i}^{i+1,i+1} E_{i+1}^* A_{i+1} E_1^* \qquad (1 \leqslant i \leqslant d-2). \tag{42}$$

In particular, for all i $(1 \leqslant i \leqslant d-1)$

$$E_i^* R^{i-1} E_1^* A E_1^* = \left(\sum_{k=1}^{i-1} \left(\prod_{j=1}^{k-1} \gamma_{j,j}^{j+1,j+1} \right) \gamma_{k,k}^{k+1,k} \left(\prod_{\ell=k}^{i-2} c_{\ell+1} \right) \right) E_i^* A_{i-1} E_1^* + \left(\prod_{j=1}^{i-1} \gamma_{j,j}^{j+1,j+1} \right) E_i^* A_i E_1^*. \tag{43}$$

Proof. To see (42), compare both sides entry-by-entry using Lemmas 3.1, 3.5, and Corollary 3.6 along with the fact that Γ is pseudo-1-homogeneous with respect to x.

Eq. (43) is proved by induction. Of course $E_1^* R^0 E_1^* A E_1^* = E_1^* A E_1^*$. Suppose that (43) holds for some i $(1 \leqslant i < d-1)$. Then $E_{i+1}^* R^i E_1^* A E_1^* = \alpha R E_i^* A_{i-1} E_1^* + \beta R E_i^* A_i E_1^*$, where α and β are the coefficients of $E_i^* A_{i-1} E_1^*$ and $E_i^* A_i E_1^*$ in (43). But $RE_i^* A_{i-1} E_1^* = c_i E_{i+1}^* A_i E_1^*$ by (32) and $RE_i^* A_i E_1^* = \gamma_{i,i}^{i+1,i} E_{i+1}^* A_i E_1^* + \gamma_{i,i}^{i+1,i+1} E_{i+1}^* A_{i+1} E_1^*$ by (42). The result now follows by induction. $\square$

Lemma 6.2. *Let Γ denote a distance-regular graph with diameter $d \geqslant 2$ and $a_1 > 0$, $a_d > 0$. Fix a base vertex x of Γ. Suppose that Γ is pseudo-1-homogeneous with respect to x, and write $\gamma_{j,k}^{h,i} = \gamma_{j,k}^{h,i}(x)$ $(0 \leqslant h \leqslant d-1,\ 0 \leqslant i,\ j,\ k \leqslant d)$. Then $\gamma_{i,i}^{i+1,i+1} \neq 0$ $(1 \leqslant i \leqslant d-2)$.*

Proof. Observe that Lemma 5.2(i) implies that $p_{i1}^i = a_i \neq 0$ $(1 \leqslant i \leqslant d)$ since $a_1 \neq 0$ and $a_d \neq 0$. Thus $p_{ii}^1 \neq 0$ $(1 \leqslant i \leqslant d)$, so for any vertex z of Γ with $\partial(x, z) = 1$, the set $\Gamma_i(x) \cap \Gamma_i(z)$ is nonempty for all i $(1 \leqslant i \leqslant d)$. In particular, there are vertices in the configurations of interest. In general, given vertices y, z of Γ with $\partial(x, y) = h$, $\partial(y, z) = i$, and $\partial(z, x) = 1$, the number $\gamma_{j,k}^{h,i}$ is the number of edges from y into $\Gamma_j(x) \cap \Gamma_k(z)$. Thus, $\gamma_{j,k}^{h,i} = 0$ if and only if $\gamma_{h,i}^{j,k} = 0$. With these observations we are ready to begin.

We proceed by induction on i beginning with the case $i = 1$. Assume for the sake of contradiction that $\gamma_{1,1}^{2,2} = 0$. Pick vertices u, z of Γ with $\partial(x, z) = 1$, $u \in \Gamma_2(x) \cap \Gamma_2(z)$. By (17) and assumption, $\gamma_{1,2}^{2,2} = c_2 > 0$, so there exists $v \in \Gamma_1(x) \cap \Gamma_1(u) \cap \Gamma_2(z)$. Now $\gamma_{2,2}^{1,1} = 0$ since $\gamma_{1,1}^{2,2} = 0$. Hence $\gamma_{2,1}^{1,1} = b_1 > 0$ by (12), so $\gamma_{1,1}(x, z, v) = \gamma_{1,1}^{1,2} > 0$. Hence there is a vertex $w \in \Gamma_1(x) \cap \Gamma_1(v) \cap \Gamma_1(z)$. In particular, $v \in \Gamma_1(x) \cap \Gamma_1(u) \cap \Gamma_1(w)$, and $\partial(u, w) = 2$ by assumption and construction. Thus $\gamma_{1,1}(x, u, w) = \gamma_{1,1}^{2,2} > 0$ as v is among the vertices counted. This contraction implies that $\gamma_{1,1}^{2,2} \neq 0$.

Now fix i $(2 \leqslant i \leqslant d-2)$, and suppose that $\gamma_{j,j}^{j+1,j+1} \neq 0$ $(1 \leqslant j < i)$. Assume for the sake of contradiction that $\gamma_{i,i}^{i+1,i+1} = 0$. Pick vertices u, z of Γ with $\partial(x, z) = 1$, $u \in \Gamma_{i+1}(x) \cap \Gamma_{i+1}(z)$. By (17) and assumption, $\gamma_{i,i+1}^{i+1,i+1} = c_{i+1} > 0$, so there exists $v \in \Gamma_i(x) \cap \Gamma_1(u) \cap \Gamma_{i+1}(z)$. Now $\gamma_{i+1,i+1}^{i,i} = 0$ since $\gamma_{i,i}^{i+1,i+1} = 0$. Hence $\gamma_{i,i+1}^{i,i} = b_i > 0$ by (12), so $\gamma_{i,i}(x, z, v) = \gamma_{i,i}^{i,i+1} > 0$. Hence there is a vertex $w \in \Gamma_i(x) \cap \Gamma_1(v) \cap \Gamma_i(z)$. Observe that there exists a path $w_1 w_2 \cdots w_i = w$ with $w_i \in \Gamma_i(x) \cap \Gamma_i(z)$ since $\gamma_{j,j}^{j+1,j+1} \neq 0$ $(1 \leqslant j \leqslant i-1)$. In particular, $v \in \Gamma_i(x) \cap \Gamma_1(u) \cap \Gamma_i(w_1)$, and $\partial(u, w_1) = i+1$. Thus $\gamma_{i,i}(x, u, w_1) = \gamma_{i,i}^{i+1,i+1} > 0$ as v is among the vertices counted. This contraction implies that $\gamma_{i,i}^{i+1,i+1} \neq 0$. $\square$

Lemma 6.3. *Let Γ denote a distance-regular graph with diameter $d \geqslant 2$. Fix a base vertex x of Γ, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant d)$. Suppose that Γ is pseudo-1-homogeneous with respect to x. If $a_1 \neq 0$, then*

$$E_d^* A E_d^* A_{d-1} E_1^* \in \operatorname{span}\left\{E_d^* A_{d-1} E_1^*,\ E_d^* J E_1^*,\ E_d^* A_{d-1} E_1^* A E_1^*\right\}.$$

Proof. Write $\gamma_{j,k}^{h,i} = \gamma_{j,k}^{h,i}(x)$ $(0 \leqslant h \leqslant d-1,\ 0 \leqslant i,\ j,\ k \leqslant d)$. Applying R to (43) at $i = d-1$ and using (32) to partially simplify, $E_d^* R^{d-1} E_1^* A E_1^* = \alpha c_{d-1} E_d^* A_{d-1} E_1^* + \beta E_d^* R E_{d-1}^* A_{d-1} E_1^*$, where α and β are the coefficients of $E_{d-1}^* A_{d-2} E_1^*$ and $E_{d-1}^* A_{d-1} E_1^*$ in (43) at $i = d-1$. Observe that $E_d^* R E_{d-1}^* A_{d-1} E_1^* = E_d^* A E_{d-1}^* A_{d-1} E_1^*$ by (7), and that $E_d^* A = E_d^* A(\sum_{i=0}^{d} E_i^*) = E_d^* A E_{d-1}^* + E_d^* A E_d^*$ by (5) and Corollary 3.3. Thus $E_d^* R E_{d-1}^* A_{d-1} E_1^* = E_d^* A A_{d-1} E_1^* - E_d^* A E_d^* A_{d-1} E_1^*$. But $A A_{d-1} = b_{d-2} A_{d-2} + a_{d-1} A_{d-1} + c_d A_d$ by (4) and $E_d^* A_{d-2} E_1^* = 0$ by Corollary 3.3. It follows that $E_d^* R^{d-1} E_1^* A E_1^* = (c_{d-1}\alpha + a_{d-1}\beta) E_d^* A_{d-1} E_1^* + c_d \beta E_d^* A_d E_1^* - \beta E_d^* A E_d^* A_{d-1} E_1^*$. Recall that $E_d^* R^{d-1} E_1^* = (c_1 c_2 \cdots c_{d-1}) E_d^* A_{d-1} E_1^*$ by (33) and that $E_d^* A_d E_1^* = E_d^* J E_1^* - E_d^* A_{d-1} E_1^*$ by (27). Thus $E_d^* A_{d-1} E_1^* A E_1^* = \gamma^{-1}(c_{d-1}\alpha + (a_{d-1} - c_d)\beta) E_d^* A_{d-1} E_1^* + c_d \beta \gamma^{-1} E_d^* J E_1^* - \beta \gamma^{-1} E_d^* A E_d^* A_{d-1} E_1^*$, where $\gamma = c_1 c_2 \cdots c_{d-1}$. If $a_d = 0$, then $E_d^* A E_d^* = 0$ by Corollary 3.3 and there is nothing more to prove. If $a_d \neq 0$, then Lemma 6.2 implies that $\beta \neq 0$ and we may solve for $E_d^* A E_d^* A_{d-1} E_1^*$ to show that it is in the desired span. $\square$

Lemma 6.4. *Let Γ denote a distance-regular graph with diameter $d \geqslant 2$. Fix a base vertex x of Γ, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant d)$. Suppose that Γ is pseudo-1-homogeneous with respect to x, and write $\gamma_{j,k}^{h,i} = \gamma_{j,k}^{h,i}(x)$ $(0 \leqslant h \leqslant d-1,\ 0 \leqslant i,\ j,\ k \leqslant d)$. Then*

$$E_1^* A E_1^* A E_1^* = (a_1 - \gamma_{1,1}^{1,2}) E_1^* + (\gamma_{1,1}^{1,1} - \gamma_{1,1}^{1,2}) E_1^* A E_1^* + \gamma_{1,1}^{1,2} E_1^* J E_1^*. \tag{44}$$

Proof. Note that $E_1^* A E_1^* A E_1^* = F E_1^* A E_1^*$, so $E_1^* A E_1^* A E_1^* = \gamma_{1,1}^{1,0} E_1^* + \gamma_{1,1}^{1,1} E_1^* A E_1^* + \gamma_{1,1}^{1,2} E_1^* A_2 E_1^*$, by (22) and Lemma 3.1 since Γ is pseudo-1-homogeneous. Observe that

$\gamma^{1,0}_{1,1} = a_1$. Also note that $E^*_1 A_2 E^*_1 = E^*_1 J E^*_1 - E^*_1 - E^*_1 A E^*_1$ by (27). The result follows. □

Proof of Theorem 1.4. (i)⇒(ii): Let U denote the linear span of $\{E^*_0 A E^*_1\} \cup \{E^*_i A_{i-1} E^*_1, E^*_i A_i E^*_1, E^*_i A_{i+1} E^*_1 \mid 1 \leqslant i \leqslant d-1\} \cup \{E^*_d A_{d-1} E^*_1, E^*_d A_d E^*_1, F E^*_d A_{d-1} E^*_1\}$. Arguing as in the Proof of Theorem 1.1 (i)⇒(ii) shows that $L E^*_i \mathcal{M} E^*_1 \subset U$ $(1 \leqslant i \leqslant d)$, $F E^*_i \mathcal{M} E^*_1 \subset U$ $(0 \leqslant i \leqslant d-1)$, $R E^*_i \mathcal{M} E^*_1 \subset U$ $(0 \leqslant i \leqslant d-2)$. By construction, $F E^*_d A_{d-1} E^*_1 \in U$, so $F E^*_d A_d E^*_1 \in U$ by (27) and (29). Furthermore, $R E^*_{d-1} A_{d-2} E^*_1 \in U$ by (32), $R E^*_{d-1} A_d E^*_1 \in U$ by (35), and $R E^*_{d-1} A_{d-1} E^*_1 \in U$ by (27) and (30). Thus it remains to show that $F F E^*_d A_{d-1} E^*_1, L F E^*_d A_{d-1} E^*_1 \in U$. Note that if $a_d = 0$, then $F E^*_d = 0$ and there is nothing further to prove. So suppose that $a_d \neq 0$.

Write $\gamma^{h,i}_{j,k} = \gamma^{h,i}_{j,k}(x)$ $(0 \leqslant h \leqslant d-1,\ 0 \leqslant i,\ j,\ k \leqslant d)$. By Lemma 6.3, there are scalars α, β, γ such that

$$F E^*_d A_{d-1} E^*_1 = \alpha E^*_d A_{d-1} E^*_1 A E^*_1 + \beta E^*_d A_{d-1} E^*_1 + \gamma E^*_d J E^*_1. \tag{45}$$

(Their values can be deduced from the proof of Lemma 6.3—in particular, $\alpha \neq 0$).

We first show that $F F E^*_d A_{d-1} E^*_1 \in U$. Observe that by (45) $F F E^*_d A_{d-1} E^*_1 = \alpha F E^*_d A_{d-1} E^*_1 A E^*_1 + \beta F E^*_d A_{d-1} E^*_1 + \gamma F E^*_d J E^*_1$. Now $F E^*_d A_{d-1} E^*_1 \in U$ by construction, and $F E^*_d J E^*_1 = a_d E^*_d J E^*_1 \in U$ by (29) and (27). The term $F E^*_d A_{d-1} E^*_1 A E^*_1$ is expanded with (45): $F E^*_d A_{d-1} E^*_1 A E^*_1 = \alpha E^*_d A_{d-1} (E^*_1 A E^*_1)^2 + \beta E^*_d A_{d-1} E^*_1 A E^*_1 + \gamma E^*_d J E^*_1 A E^*_1$. The terms in this expression are computed using (45), (29), and (44):

$$\begin{aligned}
E^*_d A_{d-1} E^*_1 A E^*_1 &= \alpha^{-1} F E^*_d A_{d-1} E^*_1 - \alpha^{-1} \beta E^*_d A_{d-1} E^*_1 - \alpha^{-1} \gamma E^*_d J E^*_1 \in U, \\
E^*_d J E^*_1 A E^*_1 &= E^*_d (F E^*_1 J)^{\mathrm{t}} = a_1 E^*_d (E^*_1 J)^{\mathrm{t}} = a_1 E^*_d J E^*_1 \in U, \\
E^*_d A_{d-1} (E^*_1 A E^*_1)^2 &= (a_1 - \gamma^{1,2}_{1,1}) E^*_d A_{d-1} E^*_1 + (\gamma^{1,1}_{1,1} - \gamma^{1,2}_{1,1}) E^*_d A_{d-1} E^*_1 A E^*_1 \\
&\quad + \gamma^{1,2}_{1,1} E^*_d A_{d-1} E^*_1 J E^*_1.
\end{aligned}$$

Finally, by (33) and (30) $E^*_d A_{d-1} E^*_1 J E^*_1 = (c_1 c_2 \cdots c_{d-1})^{-1} E^*_d R^{d-1} E^*_1 J E^*_1 = (c_d / c_1) E^*_d J E^*_1 \in U$. Thus $F F E^*_d A_{d-1} E^*_1 \in U$.

We now show that $L F E^*_d A_{d-1} E^*_1 \in U$. Observe that by (45) $L F E^*_d A_{d-1} E^*_1 = \alpha L E^*_d A_{d-1} E^*_1 A E^*_1 + \beta L E^*_d A_{d-1} E^*_1 + \gamma L E^*_d J E^*_1$. Now Γ is pseudo-1-homogeneous with respect to x, so Lemma 3.1 and Corollary 3.6 give the action of L of on $E^*_d A_{d-1} E^*_1$, while (28) gives the action of L on $E^*_d J E^*_1$:

$$\begin{aligned}
L E^*_d A_{d-1} E^*_1 &= \gamma^{d-1,d-2}_{d,d-1} E^*_{d-1} A_{d-2} E^*_1 + \gamma^{d-1,d-1}_{d,d-1} E^*_{d-1} A_{d-1} E^*_1 \\
&\quad + \gamma^{d-1,d}_{d,d-1} E^*_{d-1} A_d E^*_1 \in U, \\
L E^*_d J E^*_1 &= b_{d-1} E^*_{d-1} J E^*_1 \in U.
\end{aligned}$$

In particular, the remaining term, $L E^*_d A_{d-1} E^*_1 A E^*_1$, is a linear combination of $E^*_{d-1} A_{d-2} E^*_1 A E^*_1$, $E^*_{d-1} A_{d-1} E^*_1 A E^*_1$, and $E^*_{d-1} A_d E^*_1 A E^*_1$, so it suffices to show that each of these terms lies in U. In fact, it is enough to show that $E^*_{d-1} A_{d-1} E^*_1 A E^*_1, E^*_{d-1} A_{d-2} E^*_1 A E^*_1 \in U$ since by (29) and (27)

$$\begin{aligned}
E^*_{d-1} A_d E^*_1 A E^*_1 &= E^*_{d-1} J E^*_1 A E^*_1 - E^*_{d-1} A_{d-1} E^*_1 A E^*_1 - E^*_{d-1} A_{d-2} E^*_1 A E^*_1, \\
E^*_{d-1} J E^*_1 A E^*_1 &= E^*_{d-1} (F E^*_1 J)^{\mathrm{t}} = a_1 E^*_{d-1} (E^*_1 J)^{\mathrm{t}} = a_1 E^*_{d-1} J E^*_1 \in U.
\end{aligned}$$

Now by (33) and (43),

$$\begin{aligned}E^*_{d-1}A_{d-2}E^*_1AE^*_1 &= (c_1c_2\cdots c_{d-2})^{-1}E^*_{d-1}R^{d-2}E^*_1AE^*_1\\ &= (c_1c_2\cdots c_{d-2})^{-1}(\alpha' E^*_{d-1}A_{d-2}E^*_1+\beta' E^*_{d-1}A_{d-1}E^*_1)\in U,\end{aligned}$$

where α' and β' are the coefficients of $E^*_{d-1}A_{d-2}E^*_1$ and $E^*_{d-1}A_{d-1}E^*_1$ in (43). Note that $\beta'\neq 0$ by Lemma 6.2 since both $a_1\neq 0$ and $a_d\neq 0$. By (43) and (44)

$$\begin{aligned}E^*_{d-1}A_{d-1}E^*_1AE^*_1 &= -\alpha'(\beta')^{-1}E^*_{d-1}A_{d-2}E^*_1AE^*_1\\ &\quad +(\beta')^{-1}E^*_{d-1}R^{d-2}E^*_1AE^*_1AE^*_1,\\ E^*_{d-1}R^{d-2}E^*_1AE^*_1AE^*_1 &= (a_1-\gamma^{1,2}_{1,1})E^*_{d-1}R^{d-2}E^*_1+(\gamma^{1,1}_{1,1}-\gamma^{1,2}_{1,1})E^*_{d-1}R^{d-2}E^*_1AE^*_1\\ &\quad +\gamma^{1,2}_{1,1}E^*_{d-1}R^{d-2}E^*_1JE^*_1.\end{aligned}$$

But by (33) and (30), $E^*_{d-1}R^{d-2}E^*_1=(c_1c_2\cdots c_{d-2})E^*_{d-1}A_{d-2}E^*_1\in U$, $E^*_{d-1}R^{d-2}E^*_1AE^*_1=(c_1c_2\cdots c_{d-2})E^*_{d-1}A_{d-2}E^*_1AE^*_1\in U$, and $E^*_{d-1}R^{d-2}E^*_1JE^*_1=(c_2c_3\cdots c_{d-1})E^*_{d-1}JE^*_1\in U$. Thus $LFE^*_dA_{d-1}E^*_1\in U$. It follows that U is closed under the action of $\mathcal{T}$, so it is the minimal left ideal of $\mathcal{T}$ generated by E^*_1.

(ii)⇒(i): Observe that $E^*_hAE^*_jA_kE^*_1\in\mathcal{T}E^*_1$, so $E^*_hAE^*_jA_kE^*_1=\sum_{n=0}^d\alpha^{h,n}_{j,k}E^*_hA_nE^*_1+\beta^h_{j,k}FE^*_dA_{d-1}E^*_1$ for some scalars $\alpha^{h,n}_{j,k}=\alpha^{h,n}_{j,k}(x)$, $\beta^h_{j,k}=\beta^h_{j,k}(x)$ by (ii). Now for all vertices y, z of Γ with $\partial(x,y)=h$, $\partial(y,z)=i$, and $\partial(z,x)=1$, the (y,z)-entry of $E^*_hAE^*_jA_kE^*_1$ is $\gamma_{j,k}(x,y,z)$ by Lemma 3.5, and $(\sum_{n=0}^d\alpha^{h,n}_{j,k}E^*_hA_nE^*_1)(y,z)=\alpha^{h,i}_{j,k}$ by Lemma 3.1. Observe that $FE^*_dA_{d-1}E^*_1(y,z)=0$ unless $h=d$. Thus $\gamma_{j,k}(x,y,z)=\alpha^{h,i}_{j,k}(x)$ for all h, i, j, k $(0\leqslant h\leqslant d-1,\ 0\leqslant i,\ j,k\leqslant d)$. Hence Γ is pseudo-1-homogeneous with respect to x.

(ii)⇒(iii): Clear.

(iii)⇒(ii): Let $B=\{E^*_0AE^*_1\}\cup\{E^*_iA_{i-1}E^*_1,\ E^*_iA_iE^*_1,\ E^*_iA_{i+1}E^*_1\mid 1\leqslant i\leqslant d-1\}\cup\{E^*_dA_{d-1}E^*_1,\ E^*_dA_dE^*_1,\ FE^*_dA_{d-1}E^*_1\}$, and let U denote the linear span of B. To show that $\mathcal{T}E^*_1=U$, it is enough to show that U is closed under left multiplication by L, F, R by (6) and Lemma 2.1 since $E^*_1\in U\subseteq\mathcal{T}E^*_1$.

First, suppose $a_i=0$ for some i $(0\leqslant i\leqslant d)$. Then $FE^*_iA_{i+1}E^*_1=0$, $FE^*_iA_{i-1}E^*_1=0$, and $E^*_iA_iE^*_1=0$ by Corollary 3.3. In addition, $LE^*_iA_{i-1}E^*_1$, $LE^*_iA_{i+1}E^*_1$, $RE^*_iA_{i+1}E^*_1$, $RE^*_iA_{i-1}E^*_1\in U$ by (31), (32), (40), and (41). In the case $i=d$, many of these expressions are zero, but the argument still proceeds as above.

Now suppose $a_i\neq 0$ for some i $(1\leqslant i\leqslant d-1)$. Then $E^*_i\mathcal{T}E^*_1$ has linear basis $\{E^*_iA_{i-1}E^*_1,\ E^*_iA_iE^*_1,\ E^*_iA_{i+1}E^*_1\}$ by Corollaries 3.2 and 3.3 and since $\dim E^*_i\mathcal{T}E^*_1\leqslant 3$ by assumption. Thus it must be the case that $FE^*_iA_{i-1}E^*_1$, $FE^*_iA_iE^*_1$, and $FE^*_iA_{i+1}E^*_1\in U$. If $a_{i-1}=0$, then $LE^*_iA_{i-1}E^*_1$, $LE^*_iA_iE^*_1$, $LE^*_iA_{i+1}E^*_1\in U$ by (36), (37), and (31). If $a_{i-1}\neq 0$, then $E^*_{i-1}\mathcal{T}E^*_1=\operatorname{span}\{E^*_{i-1}A_{i-2}E^*_1,\ E^*_{i-1}A_{i-1}E^*_1,\ E^*_{i-1}A_iE^*_1\}$, so it must be the case that $LE^*_iA_{i-1}E^*_1$, $LE^*_iA_iE^*_1$, $LE^*_iA_{i+1}E^*_1\in U$. If $a_{i+1}=0$, then $RE^*_iA_{i-1}E^*_1$, $RE^*_iA_iE^*_1$, $RE^*_iA_{i+1}E^*_1\in U$ by (32), (38), and (39). If $i<d-1$ and $a_{i+1}\neq 0$, then by the above $E^*_{i+1}\mathcal{T}E^*_1=\operatorname{span}\{E^*_{i+1}A_iE^*_1,\ E^*_{i+1}A_{i+1}E^*_1,\ E^*_{i+1}A_{i+2}E^*_1\}$, so it must be the case that $RE^*_iA_{i-1}E^*_1$, $RE^*_iA_iE^*_1$, $RE^*_iA_{i+1}E^*_1\in U$. In addition, $RE^*_{d-1}A_{d-2}E^*_1$, $RE^*_{d-1}A_{d-1}E^*_1$, $RE^*_{d-1}A_dE^*_1\in\operatorname{span}\{E^*_dA_{d-1}E^*_1,\ E^*_dA_dE^*_1,\ FE^*_dE^*_{d-1}E^*_1\}\subseteq U$ by (30), (27), (32), and (35).

Finally, suppose $a_d \neq 0$. Naturally $RE^*_d A_{d-1} E^*_1 = 0$, $RE^*_d A_d E^*_1 = 0$, $RFE^*_d A_{d-1} E^*_1 = 0$. If $FE^*_d A_{d-1} E^*_1 \in \operatorname{span}\left\{E^*_d A_{d-1} E^*_1, E^*_d A_d E^*_1\right\}$, then $FE^*_d A_d E^*_1$, $FFE^*_d A_{d-1} E^*_1 \in U$ by (27), (29), and construction. If $FE^*_d A_{d-1} E^*_1 \notin \operatorname{span}\left\{E^*_d A_{d-1} E^*_1, E^*_d A_d E^*_1\right\}$, then $E^*_d \mathcal{T} E^*_1$ has basis $E^*_d A_{d-1} E^*_1$, $E^*_d A_d E^*_1$, $FE^*_d A_{d-1} E^*_1$ since $\dim E^*_d \mathcal{T} E^*_1 \leqslant 3$, so it must be the case that $FE^*_d A_d E^*_1$, $FF \in E^*_d A_{d-1} E^*_1 \in U$. If $a_{d-1} = 0$, then $FE^*_d A_{d-1} E^*_1 \in \operatorname{span}\left\{E^*_d A_{d-1} E^*_1, E^*_d A_d E^*_1\right\}$ by (35) and (41), $LE^*_d A_{d-1} E^*_1 \in U$ by (36), and $LE^*_d A_d E^*_1 \in U$ by (29) and (27). If $a_{d-1} \neq 0$, then $LE^*_d A_{d-1} E^*_1$ and $LE^*_d A_d E^*_1$, $LFE^*_d A_{d-1} E^*_1 \in U$ since $E^*_{d-1} \mathcal{T} E^*_1$ has a basis consisting of elements of B by the above. In any case, U is closed under left multiplication by L, F, and R, so (ii) holds. □

Corollary 6.5. *Let Γ denote a distance-regular graph with diameter $d \geqslant 2$. Fix a base vertex x of Γ, and write $E^*_i = E^*_i(x)$ $(0 \leqslant i \leqslant d)$ and $\mathcal{T} = \mathcal{T}(x)$. Suppose that $a_1 \neq 0$ and that Γ is pseudo-1-homogeneous but not 1-homogeneous with respect to x. Then the following hold.*

(i) *$E^*_0 \mathcal{T} E^*_1$ has linear basis $\{E^*_0 A E^*_1\}$.*
(ii) *For $1 \leqslant i \leqslant d-1$, $E^*_i \mathcal{T} E^*_1$ has linear basis $\{E^*_i A_{i-1} E^*_1, E^*_i A_i E^*_1, E^*_i A_{i+1} E^*_1\}$.*
(iii) *$E^*_d \mathcal{T} E^*_1$ has linear basis $\{E^*_d A_{d-1} E^*_1, E^*_d A_d E^*_1, FE^*_d A_{d-1} E^*_1\}$.*

Proof. Observe that $E^*_0 \mathcal{T} E^*_1$ has linear basis $\{E^*_0 A E^*_1\}$ by Theorem 1.4. Since Γ is not 1-homogeneous with respect to x, we must have $\dim E^*_d \mathcal{T} E^*_1 = 3$ by Theorems 1.1 and 1.4, so $\{E^*_d A_{d-1} E^*_1, E^*_d A_d E^*_1, FE^*_d A_{d-1} E^*_1\}$ form a basis $E^*_d \mathcal{T} E^*_1$. In particular, $a_d \neq 0$. By Theorem 1.4, $E^*_i \mathcal{T} E^*_1$ is spanned by $E^*_i A_{i-1} E^*_1$, $E^*_i A_i E^*_1$, and $E^*_i A_{i+1} E^*_1$ $(1 \leqslant i \leqslant d-1)$. By Corollaries 3.2 and 3.3, these three matrices are linearly independent when $a_i \neq 0$. Observe that it is the case that $a_i \neq 0$ by Lemma 5.2 since $a_1 \neq 0$ and $a_d \neq 0$. □

7. The decomposition of $\mathcal{T} E^*_1$

Our goal in this section is to describe the direct sum decomposition of the left ideal $\mathcal{T} E^*_1$ into minimal left ideals of $\mathcal{T}$ for the 1-homogeneous and pseudo-1-homogeneous distance-regular graphs. Let us begin with some results which assure us that such a decomposition exists.

Lemma 7.1 (*Terwilliger [25]*). *Let Γ denote a distance-regular graph with diameter d. Fix a base vertex x of Γ, and write $E^*_i = E^*_i(x)$ $(0 \leqslant i \leqslant d)$ and $\mathcal{T} = \mathcal{T}(x)$. Then $\mathcal{T}$ is a complex finite-dimensional semisimple algebra.*

Proof. Observe that $\mathcal{T}$ is a subalgebra of $Mat_X(\mathbb{C})$ so it is a finite-dimensional complex algebra. Recall that $\mathcal{T}$ is generated by A (the adjacency matrix of Γ), $E^*_0, \ldots, E^*_d$. These are real and symmetric matrices, so $\mathcal{T}$ is closed under the conjugate-transpose operation. It follows that $\mathcal{T}$ is semisimple. □

Complex finite-dimensional semisimple algebras are described by Wedderburn theory. They have a particularly nice structure.

Lemma 7.2 (*Wedderburn, see [12]*). *Let $\mathcal{S}$ denote a finite-dimensional complex semisimple algebra.*

(i) *$\mathcal{S}$ is isomorphic to a direct sum of complex matrix algebras.*
(ii) *The center of $\mathcal{S}$ has a basis consisting of primitive central idempotents.*
(iii) *Each primitive central idempotent generates a minimal two-sided ideal of $\mathcal{S}$ and every minimal two-sided ideal of $\mathcal{S}$ is generated by a primitive central idempotent.*
(iv) *$\mathcal{S}$ is completely reducible. That is, every two-sided* (*respectively left*) *ideal of $\mathcal{S}$ decomposes into a direct sum of minimal two-sided* (*respectively left*) *ideals.*

The decomposition of $\mathcal{T}$ into minimal two-sided ideals is unique but this is not the case for minimal left ideals. However, every minimal left ideal of $\mathcal{T}$ is contained entirely within a minimal two-sided ideal of $\mathcal{T}$ (multiplying any left ideal by a primitive central idempotent produces another left ideal contained entirely within one minimal two-sided ideal). Moreover, each minimal left ideal of $\mathcal{T}$ contained in a given minimal two-sided ideal has the same dimension. The following lemma describes a minimal two-sided ideal and a decomposition into minimal left ideals.

Lemma 7.3 (*Terwilliger [25]*). *Let Γ denote a distance-regular graph with diameter d. Fix any vertex x of Γ, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant d)$ and $\mathcal{T} = \mathcal{T}(x)$. The two-sided ideal of $\mathcal{T}$ generated by J is minimal and has linear basis $\{E_i^* J E_j^* \mid 0 \leqslant i,\ j \leqslant d\}$. This two-sided ideal has a direct sum decomposition into $d+1$ many minimal left ideals $\mathcal{T} J E_j^*$ $(0 \leqslant i \leqslant d)$. Moreover, for each j, $\{E_i^* J E_j^* \mid 0 \leqslant i \leqslant d\}$ is a linear basis for $\mathcal{T} J E_j^*$.*

Proof. Let U denote the linear span of $B = \{E_i^* J E_j^* \mid 0 \leqslant i,\ j \leqslant d\}$. Now U is closed under left multiplication by $E_0^*, E_1^*, \ldots, E_d^*$ by (6) and U is closed under left multiplication by L, F, and R by (28)–(30). Thus U is a left ideal by Lemma 2.1. Similarly, U is a right ideal since $R^{\mathrm{t}} = L$, $F^{\mathrm{t}} = F$, $E_i^{*\mathrm{t}} = E_i^*$, and $J^{\mathrm{t}} = J$. Now (28)–(30) imply that U is a minimal two-sided ideal. Observe that $J \in U$ by (5), so U is equal to the two-sided ideal generated by J. Finally, observe that B is linearly independent since no two of theses matrices have a common nonzero entry. The assertions about the left ideals $\mathcal{T} J E_j^*$ can be shown in a similar way. $\square$

To further describe the decomposition of $\mathcal{T}$ into minimal left ideals, we shall consider the left multiplication of the generators $L, F, R, E_0^*, E_1^*, \ldots, E_d^*$ of $\mathcal{T}$ on bases of the left ideals of $\mathcal{T}$. First we need some notation.

Lemma 7.4 (*Terwilliger[25]*). *Let Γ denote a distance-regular graph with diameter d. Fix a base vertex x of Γ, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant d)$ and $\mathcal{T} = \mathcal{T}(x)$. Let $\mathcal{L}$ denote a minimal left ideal of $\mathcal{T}$. Then there exist integers r and ℓ such that $E_i^* \mathcal{L} \neq 0$ if and only if $r \leqslant i \leqslant r+\ell$.*

Definition 7.5. Let Γ denote a distance-regular graph with diameter d. Fix a base vertex x of Γ, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant d)$ and $\mathcal{T} = \mathcal{T}(x)$. Let $\mathcal{L}$ denote a minimal left ideal of $\mathcal{T}$. The numbers r and ℓ of Lemma 7.4 are called the *endpoint* and *diameter* of $\mathcal{L}$, respectively. We say that $\mathcal{L}$ is *thin* if $\dim E_i^* \mathcal{L} \leqslant 1$ $(0 \leqslant i \leqslant d)$.

In the previous section we saw that the 1-homogeneous property was characterized by a condition on $\mathcal{T}E_1^*$. Thus we shall focus on the minimal left ideals of $\mathcal{T}$ contained in $\mathcal{T}E_1^*$. We show that there is a unique such left ideal with endpoint 0, and that this left ideal is thin and has diameter d. We then show that any other minimal left ideal of $\mathcal{T}$ contained in $\mathcal{T}E_1^*$ has endpoint 1 and diameter $d-2$ or $d-1$.

Lemma 7.6. *Let Γ denote a distance-regular graph with diameter d. Fix a base vertex x of Γ, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant d)$ and $\mathcal{T} = \mathcal{T}(x)$. Then $\mathcal{T}JE_1^*$ is a thin minimal left ideal of $\mathcal{T}$ contained in $\mathcal{T}E_1^*$ with endpoint* 0 *and diameter d. Moreover, $\mathcal{T}JE_1^*$ is the unique minimal left ideal contained in $\mathcal{T}E_1^*$ with endpoint* 0. *We refer to this left ideal as the* trivial minimal left ideal contained in $\mathcal{T}E_1^*$.

Proof. Clear since $\mathcal{T}JE_1^*$ has linear basis $\{E_i^*JE_1^* \mid 0 \leqslant i \leqslant d\}$ by Lemma 7.3. □

Lemma 7.7. *Let Γ denote a distance-regular graph with diameter d. Fix a base vertex x of Γ, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant d)$ and $\mathcal{T} = \mathcal{T}(x)$. Let $\mathcal{L}$ denote a minimal left ideal of $\mathcal{T}$ contained in $\mathcal{T}E_1^*$. If $\mathcal{L}$ has endpoint greater than zero, then $J\mathcal{L} = 0$.*

Proof. By Lemma 7.3, J generates a minimal two-sided ideal of $\mathcal{T}$ which decomposes into a direct sum of minimal left ideals $\mathcal{T}JE_j^*$. All of these minimal left ideals have endpoint zero since $E_0^*JE_j^*$ is a nonzero element and by (6). But $J\mathcal{L}$ must be contained in the two-sided ideal generated by J and in the minimal left ideal $\mathcal{L}$. Since $\mathcal{L}$ is contained in $\mathcal{T}E_1^*$, it must lie entirely in the direct summand $\mathcal{T}JE_1^*$. The minimality of $\mathcal{L}$ and $\mathcal{T}JE_1^*$ force that either $\mathcal{L} = \mathcal{T}JE_1^*$ or $J\mathcal{L} = 0$. Equality cannot hold since $\mathcal{L}$ has endpoint greater than zero. Thus $J\mathcal{L} = 0$. □

Lemma 7.8. *Let Γ denote a distance-regular graph with diameter d. Fix a base vertex x of Γ, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant d)$ and $\mathcal{T} = \mathcal{T}(x)$. Let $\mathcal{L}$ denote a minimal left ideal of $\mathcal{T}$ contained in $\mathcal{T}E_1^*$. Then $E_1^*\mathcal{L} \neq 0$, ie, $\mathcal{L}$ has endpoint* 0 *or* 1.

Proof. Suppose $E_1^*\mathcal{L} = 0$. Observe that $\mathcal{I} = \mathcal{L}\mathcal{T}$ is the minimal two-sided ideal of $\mathcal{T}$ which contains $\mathcal{L}$, so that $\mathcal{I}$ contains a unique primitive central primitive idempotent φ of $\mathcal{T}$ which behaves as the identity on $\mathcal{I}$. Now $E_1^*\varphi \in E_1^*\mathcal{I} = E_1^*\mathcal{L}\mathcal{T} = 0$, so $E_1^*\varphi = 0$. But $\mathcal{L}E_1^* = \mathcal{L}$ since $\mathcal{L} \subseteq \mathcal{T}E_1^*$. This implies that $\mathcal{L} = \mathcal{L}E_1^* = \mathcal{L}E_1^*\varphi = 0$. □

Lemma 7.9. *Let Γ denote a distance-regular graph with diameter $d \geqslant 2$. Fix a base vertex x of Γ, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant d)$ and $\mathcal{T} = \mathcal{T}(x)$. Let $\mathcal{L}$ denote any nontrivial minimal left ideal of $\mathcal{T}$ contained in $\mathcal{T}E_1^*$. Then $E_i^*\mathcal{L} \neq 0$ $(1 \leqslant i \leqslant d-1)$, ie, $\mathcal{L}$ has diameter $d-2$ or $d-1$.*

Proof. Suppose $E_{d-1}^*\mathcal{L} = 0$. Then $E_{d-1}^*A_dE_1^*\mathcal{L} = 0$. But by (31) $L^jE_{d-1}^*A_dE_1^* = (b_{d-1}b_{d-2}\cdots b_{d-1-j})E_{d-j-1}^*A_{d-j}E_1^*$. Thus $E_\ell^*A_{\ell+1}E_1^*\mathcal{L} = 0$ $(0 \le \ell \leqslant d-1)$. Observe that (34) reduces to $(LE_{\ell+1}^*A_\ell E_1^* - b_\ell E_\ell^*A_{\ell-1}E_1^*)\mathcal{L} = 0$ $(1 \leqslant \ell \leqslant d-1)$, and this implies that $(L^{d-2}E_{d-1}^*A_{d-2}E_1^* - (b_{d-2}b_{d-3}\cdots b_1)E_1^*A_0E_1^*)\mathcal{L} = 0$. But this is impossible: $L^{d-2}E_{d-1}^*A_{d-2}E_1^*\mathcal{L} = 0$ by $E_{d-1}^*\mathcal{L} = 0$ and $(b_{d-2}b_{d-3}\cdots b_1)E_1^*\mathcal{L} \neq 0$ by Lemma 7.8. Thus $E_{d-1}^*\mathcal{L} \neq 0$. □

Our next goal is to relate the minimal left ideals of $\mathcal{T}$ contained in $\mathcal{T}E_1^*$ to the space $E_1^*\mathcal{T}E_1^*$.

Lemma 7.10. *Let Γ denote a distance-regular graph with diameter d. Fix a base vertex x of Γ, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant d)$ and $\mathcal{T} = \mathcal{T}(x)$. Then $E_1^*\mathcal{T}E_1^*$ is a semisimple subalgebra of $Mat_X(\mathbb{C})$.*

Proof. Observe that by Lemma 2.1, every element of $E_1^*\mathcal{T}E_1^*$ is a linear combination of products of L, F, R with L and R appearing an equal number of times and multiplied on both sides by E_1^*. But all of these matrices are real and $L^{\mathrm{t}} = R$, $F^{\mathrm{t}} = F$. Thus $E_1^*\mathcal{T}E_1^*$ is closed under conjugate-transpose, so it is semisimple. □

Lemma 7.11. *Let Γ denote a distance-regular graph with diameter d. Fix a base vertex x of Γ, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant d)$ and $\mathcal{T} = \mathcal{T}(x)$. Let $\psi_1, \psi_2, \ldots, \psi_s$ denote the primitive central idempotents of $\mathcal{T}$. Then the set of primitive central idempotents of $E_1^*\mathcal{T}E_1^*$ consists of all nonzero expressions $E_1^*\psi_i E_1^*$ $(1 \leqslant i \leqslant s)$.*

Proof. Since $E_1^*\mathcal{T}E_1^*$ is semisimple, its center has a basis of primitive central idempotents $\varphi_1, \varphi_2, \ldots, \varphi_r$. Let $\psi_1, \psi_2, \ldots, \psi_s$ denote the primitive central idempotents of $\mathcal{T}$, and set $\psi_i' = E_1^*\psi_i E_1^*$ $(1 \leqslant i \leqslant s)$. First, observe that ψ_i' lies in the center of $E_1^*\mathcal{T}E_1^*$ since ψ_i lies in the center of $\mathcal{T}$ and $E_1^{*2} = E_1^*$. Thus each ψ_i' is a linear combination of $\varphi_1, \varphi_2, \ldots, \varphi_r$. Second, observe that ψ_i' is an idempotent. Thus each ψ_i' is the sum of some of the φ_j. Third, observe that $\psi_i'\psi_j' = \delta_{ij}$ by the primitivity of ψ_i and ψ_j. Thus each φ_i is a summand in at most one of the ψ_i'. Finally, observe that $\sum_{i=1}^{s}\psi_i' = E_1^*$ since $\sum_{i=0}^{s}\psi_i = I$. Thus each φ_i appears as a summand in the expression of some ψ_i'.

Now suppose $\psi_i' \neq 0$ is given by $\psi_i' = \varphi_{i_1} + \varphi_{i_2} + \cdots + \varphi_{i_n}$. Then ψ_i and φ_{i_j} $(1 \leqslant j \leqslant n)$ generate the same minimal two-sided ideal $\psi_i\mathcal{T}$ of $\mathcal{T}$ since all are contained in this minimal two-sided ideal by construction (here observe that $\varphi_{i_j} = \varphi_{i_j}\psi_i' \in \mathcal{T}\psi_i' = \mathcal{T}\psi_i$). Thus there exists some $t', t'' \in \mathcal{T}$ such that $\varphi_{i_2} = t'\varphi_{i_1}t''$. Observe that $\varphi_{i_j} = \varphi_{i_j}E_1^* = E_1^*\varphi_{i_j}$ for all j since $\varphi_{i_j} \in E_1^*\mathcal{T}E_1^*$. Thus we may take $t', t'' \in E_1^*\mathcal{T}E_1^*$, in which case they commute with φ_{i_1}. So in fact, $\varphi_{i_2} = t\varphi_{i_1}$ for some $t \in E_1^*\mathcal{T}E_1^*$. But now multiplying each side of this expression by φ_{i_2} gives $0 = t\varphi_{i_1}\varphi_{i_2} = \varphi_{i_2}^2 = \varphi_{i_2}$. It follows that each nonzero term ψ_i' is equal to a unique φ_j and that each φ_j is equal to some ψ_i'. □

Lemma 7.12. *Let Γ denote a distance-regular graph with diameter d. Fix a base vertex x of Γ, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant d)$ and $\mathcal{T} = \mathcal{T}(x)$. Let $\mathcal{L}$ denote a left ideal of $\mathcal{T}$. If there exists an element $G \in E_1^*\mathcal{L}$ such that G generates $\mathcal{L}$ as a left ideal of $\mathcal{T}$ and G spans $E_1^*\mathcal{L}$, then $\mathcal{L}$ is a minimal left ideal of $\mathcal{T}$.*

Proof. By complete reducibility, $\mathcal{L}$ contains a minimal left ideal $\mathcal{L}'$ such that $E_1^*\mathcal{L}' \neq 0$. Since $E_1^*\mathcal{L}'$ is contained in $E_1^*\mathcal{L}$, it must be the case that $E_1^*\mathcal{L}' = E_1^*\mathcal{L}$ since $\dim E_1^*\mathcal{L} = 1$. Thus $G \in \mathcal{L}'$. But G generates $\mathcal{L}$, so $\mathcal{L} \subseteq \mathcal{L}'$. Thus $\mathcal{L} = \mathcal{L}'$ is minimal. □

Lemma 7.13. *Let Γ denote a distance-regular graph with diameter $d \geqslant 2$. Fix a base vertex x of Γ, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant d)$ and $\mathcal{T} = \mathcal{T}(x)$. If $E_1^* \mathcal{T} E_1^*$ is commutative, then the left ideals of $\mathcal{T}$ generated by the primitive central idempotents of $E_1^* \mathcal{T} E_1^*$ are the summands in a direct sum decomposition of $\mathcal{T} E_1^*$ into minimal left ideals.*

Proof. Since $E_1^* \mathcal{T} E_1^*$ is commutative and semisimple by Lemma 7.10, its primitive central idempotents form a linear basis. Let φ denote a primitive central idempotent of $E_1^* \mathcal{T} E_1^*$, and consider the left ideal $\mathcal{L}$ of $\mathcal{T}$ generated by φ. Clearly $\mathcal{L} \subset \mathcal{T} E_1^*$, so $E_1^* \mathcal{L} \subseteq E_1^* \mathcal{T} E_1^*$. Now $E_1^* \mathcal{L}$ is spanned by φ since all other primitive central idempotents of $E_1^* \mathcal{T} E_1^*$ vanish on $E_1^* \mathcal{L} \subseteq \varphi E_1^* \mathcal{T} E_1^*$. Thus $\mathcal{L}$ is minimal by Lemma 7.12. Since the primitive central idempotents of $E_1^* \mathcal{T} E_1^*$ form a linear basis for $E_1^* \mathcal{T} E_1^*$, Lemma 7.8 implies that this accounts for all minimal left ideals contained in $\mathcal{T} E_1^*$. □

Terwilliger [25], showed that $E_1^* \mathcal{T} E_1^*$ is commutative when every minimal left ideal of endpoint one is thin. (In fact, Terwilliger showed that $E_1^* \mathcal{T} E_1^*$ is generated by $E_1^* J E_1^*$ and $E_1^* A E_1^*$ in this case). More information about this can also be found in [26,27]. We do not need to assume that any minimal left ideals of $\mathcal{T}$ are thin because of the following observation.

Lemma 7.14. *Let Γ denote a distance-regular graph with diameter $d \geqslant 2$. Fix a base vertex x of Γ, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant d)$ and $\mathcal{T} = \mathcal{T}(x)$. Then the following hold.*
(i) *If* $\dim E_1^* \mathcal{T} E_1^* = 2$, *then $E_1^* \mathcal{T} E_1^*$ is commutative and symmetric.*
(ii) *If $a_1 > 0$ and* $\dim E_1^* \mathcal{T} E_1^* = 3$, *then $E_1^* \mathcal{T} E_1^*$ is commutative and symmetric.*

Proof. For convenience, set $\tilde{I} = E_1^*$, $\tilde{J} = E_1^* J E_1^*$, and $\tilde{A} = E_1^* A E_1^*$.

(i) Observe that $\tilde{I}$ and $\tilde{J} \in E_1^* \mathcal{T} E_1^*$ are linearly independent since $d \geqslant 2$ implies that there is more than one vertex adjacent to the base point x. These two matrices are clearly symmetric and commute with one another.

(ii) If $\tilde{A}$ is in the linear span of $\tilde{I}$ and $\tilde{J}$, then either $\tilde{A} = 0$ or the induced graph on $\Gamma_1(x)$ is a clique (since $\tilde{A}$ is a zero-one matrix with zeros on the diagonal). In the former case $a_1 = 0$ by Corollary 3.3. In the latter case, $b_0 = a_1 + 1 = a_1 + c_1$, forcing $b_1 = 0$, so $d = 1$. This contradicts the assumption that $d \geqslant 2$.

Now $\tilde{I}$, $\tilde{J}$, and $\tilde{A}$ form a linear basis of $E_1^* \mathcal{T} E_1^*$. All three of these matrices are symmetric. Observe that $\tilde{I}$ acts as the identity on elements of $E_1^* \mathcal{T} E_1^*$. Also $\tilde{A}\tilde{J} = \tilde{J}\tilde{A} = a_1 \tilde{J}$. Thus $E_1^* \mathcal{T} E_1^*$ is commutative. □

8. Proofs of main results

We now prove the remaining theorems stated in the introduction.

Proof of Theorem 1.2. (i), (ii) That every bipartite and almost bipartite distance-regular graph is 1-homogeneous was shown in Lemma 5.3.

(i)–(iv) First suppose that Γ is 1-homogeneous with respect to x. Observe that $a_1 = 0$ in cases (i)–(iv), so $\dim E_1^* \mathcal{T} E_1^* = 2$ by Corollary 5.1. Thus $\mathcal{T} E_1^* = \mathcal{T} J E_1^* \oplus \mathcal{L}$, where $\mathcal{L}$

is a minimal left ideal of $\mathcal{T}$ with endpoint 1 by Lemmas 7.8, 7.13, and 7.14. By Lemma 7.6, dim $E_i^*\mathcal{T}JE_1^*=1$ $(0\leqslant i\leqslant d)$. Hence dim $E_i^*\mathcal{L}=$ dim $E_i^*\mathcal{T}E_1^*-1$ $(0\leqslant i\leqslant d)$. But for $0\leqslant i\leqslant d-1$, dim $E_i^*\mathcal{T}E_1^*=2$ if $a_i=0$ and dim $E_i^*\mathcal{T}E_1^*=3$ if $a_i\neq 0$ by Corollary 5.1. Similarly dim $E_d^*\mathcal{T}E_1^*=1$ if $a_d=0$ and dim $E_d^*\mathcal{T}E_1^*=2$ if $a_d\neq 0$. The description of which a_i may vanish given in Lemma 5.2 implies the minimal left ideal descriptions of conditions (i)–(iv).

Conversely, in each of the cases (i)–(iv), $\mathcal{T}E_1^*=\mathcal{T}JE_1^*\oplus\mathcal{L}$, where $\mathcal{L}$ is a minimal left ideal of $\mathcal{T}$ with endpoint 1. Thus dim $E_i^*\mathcal{T}E_1^*=$ dim $E_i^*\mathcal{T}JE_1^*+$ dim $E_i^*\mathcal{L}$. But dim $E_i^*\mathcal{T}JE_1^*=1$ $(0\leqslant i\leqslant d)$ by Lemma 7.3, and by assumption dim $E_i^*\mathcal{L}\leqslant 2$ $(1\leqslant i\leqslant d-1)$, dim $E_d^*\mathcal{L}\leqslant 1$. Thus dim $E_i^*\mathcal{T}E_1^*\leqslant 3$ $(1\leqslant i\leqslant d-1)$ and dim $E_d^*\mathcal{T}E_1^*\leqslant 2$. Hence Γ is 1-homogeneous with respect to x by Theorem 1.1. Moreover, $a_i=0$ if and only if dim $E_i^*\mathcal{T}E_1^*=2$ $(1\leqslant i\leqslant d-1)$ and $a_d=0$ if and only if dim $E_d^*\mathcal{T}E_1^*=1$ by Corollary 5.1.

(v), (vi) First suppose that Γ is 1-homogeneous with respect to x. Observe that $a_1\neq 0$ in cases (v), (vi), so dim $E_1^*\mathcal{T}E_1^*=3$ by Corollary 5.1. Thus $\mathcal{T}E_1^*=\mathcal{T}JE_1^*\oplus\mathcal{L}_1\oplus\mathcal{L}_1$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are minimal left ideals of $\mathcal{T}$ with endpoint 1 by Lemmas 7.8, 7.13, and 7.14. By Lemma 7.6, dim $E_i^*\mathcal{T}JE_1^*=1$ $(0\leqslant i\leqslant d)$. By Lemma 7.9, dim $E_i^*\mathcal{L}_jE_1^*\geqslant 1$ $(1\leqslant i\leqslant d-1)$ for $j=1,\ 2$. This forces dim $E_i^*\mathcal{L}_jE_1^*=1$ $(1\leqslant i\leqslant d-1)$ for $j=1,\ 2$ since dim $E_i^*\mathcal{T}E_1^*\leqslant 3$ $(1\leqslant i\leqslant d-1)$ by Theorem 1.1. If $a_d=0$, then dim $E_d^*\mathcal{T}E_1^*=1$ by Corollary 5.1, so dim $E_d^*\mathcal{L}_1=0$ and dim $E_d^*\mathcal{L}_2=0$. If $a_d\neq 0$, then dim $E_d^*\mathcal{T}E_1^*=2$, so one of $\mathcal{L}_1$, $\mathcal{L}_2$ has diameter $d-1$ and the other has diameter $d-2$.

Conversely, in each of (v), (vi), $\mathcal{T}E_1^*=\mathcal{T}JE_1^*\oplus\mathcal{L}_1\oplus\mathcal{L}_1$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are thin minimal left ideals of $\mathcal{T}$ with endpoint 1. Thus dim $E_i^*\mathcal{T}E_1^*\leqslant 3$ $(1\leqslant i\leqslant d-1)$. Furthermore, at most one of $\mathcal{L}_1$ and $\mathcal{L}_2$ has diameter $d-1$, so dim $E_d^*\mathcal{T}E_1^*\leqslant 2$. Thus Γ is 1-homogeneous with respect to x by Theorem 1.1. Furthermore, $a_d=0$ if and only if dim $E_d^*\mathcal{T}E_1^*=1$ by Corollary 5.1, so $a_d=0$ in case (v) but not in case (vi). □

Proof of Theorem 1.3. Suppose $\mathcal{T}E_1^*=\mathcal{T}JE_1^*\oplus\mathcal{L}$, where $\mathcal{L}$ is a thin minimal left ideal of $\mathcal{T}$ with endpoint 1. By Lemma 7.9 $\mathcal{L}$ has diameter $d-1$ or $d-2$. By Theorem 1.2 these cases correspond to Γ bipartite or almost bipartite.

Conversely, suppose Γ is bipartite or almost bipartite. Then $\mathcal{T}E_1^*=\mathcal{T}JE_1^*\oplus\mathcal{L}$, where $\mathcal{L}$ is a thin minimal left ideal of $\mathcal{T}$ with endpoint 1 by Theorem 1.2. □

Proof of Theorem 1.5. First suppose that Γ is pseudo-1-homogeneous with respect to x but not 1-homogeneous with respect to x. Arguing as in the proof of Theorem 1.2 (v), (vi), we find that $\mathcal{T}E_1^*=\mathcal{T}JE_1^*\oplus\mathcal{L}_1\oplus\mathcal{L}_1$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are minimal left ideals of $\mathcal{T}$ with endpoint 1 by Lemmas 7.8, 7.13, and 7.14. Moreover, dim $E_i^*\mathcal{L}_\ell=1$ $(1\leqslant i\leqslant d-1)$ for $\ell=1,\ 2$. Let φ_ℓ be any nonzero element of $E_1^*\mathcal{L}_\ell$ $(\ell=1,\ 2)$. Then by (32) $E_i^*\mathcal{L}_\ell=\text{span}\left\{E_i^*A_{i-1}E_1^*\varphi_\ell\right\}$ $(1\leqslant i\leqslant d-1)$. Now $RE_{d-1}^*\mathcal{L}_\ell=\text{span}\left\{E_d^*A_{d-1}E_1^*\varphi_\ell\right\}$ by (32). Thus $FE_d^*A_{d-1}E_1^*\varphi_\ell\in\text{span}\left\{E_d^*A_{d-1}E_1^*\varphi_\ell\right\}$ by (35). Thus the span of $\{E_i^*A_{i-1}E_1^*\varphi_\ell|1\leqslant i\leqslant d\}$ is a left ideal of $\mathcal{T}$, so that it coincides with $\mathcal{L}_\ell$ by minimality of $\mathcal{L}_\ell$ $(\ell=1,2)$. This implies dim $E_d^*\mathcal{L}_\ell\leqslant 1$ $(\ell=1,2)$. But dim $E_d^*\mathcal{T}E_1^*=3$ forces equality. Hence both $\mathcal{L}_1$, $\mathcal{L}_2$ have diameter $d-1$ and are thin.

Conversely, if $\mathcal{T}E_1^*=\mathcal{T}JE_1^*\oplus\mathcal{L}_1\oplus\mathcal{L}_1$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are thin minimal left ideals of $\mathcal{T}$ with endpoint 1 and diameter $d-1$, then dim $E_i^*\mathcal{T}E_1^*=3$ $(1\leqslant i\leqslant d)$. Thus Γ is pseudo-1-homogeneous with respect to x by Theorem 1.4. □

Proof of Theorem 1.6. Suppose $\mathcal{T}E_1^* = \mathcal{T}JE_1^* \oplus \mathcal{L}_1 \oplus \mathcal{L}_2$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are thin minimal left ideals of $\mathcal{T}$ with endpoint 1. By Lemma 7.9 each of $\mathcal{L}_1$ and $\mathcal{L}_1$ has diameter $d-1$ or $d-2$. By Theorems 1.2 and 1.5 all cases correspond to the case that Γ is pseudo-1-homogeneous with respect to x with $a_1 \neq 0$.

Conversely, suppose Γ is pseudo-1-homogeneous with respect to x with $a_1 \neq 0$. Then $\mathcal{T}E_1^* = \mathcal{T}JE_1^* \oplus \mathcal{L}_1 \oplus \mathcal{L}_2$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are thin minimal left ideals of $\mathcal{T}$ with endpoint 1 by Theorems 1.2 and 1.5. □

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